Chapter 5: Coordination and Externalities in Macroeconomics

I. Motivational Questions and Exercises:

Exercise 5.1:

(a) Derive the elasticity $\varepsilon(eb(e)) = 1 + \frac{eb'(e)}{b(e)}$ given on p. 173 of the textbook.
(b) Derive equation (5.6) on p. 175 of the textbook.
(c) Demonstrate $\lim_{\Delta t \to 0} \frac{e^{-r\Delta t}\left(1 - e^{-b\Delta t}\right)}{1 - e^{-(r+b)\Delta t}} = \frac{b}{r+b}$ given on p. 175 of the textbook.

Solutions:

Subquestion (a): By definition, the elasticity of $eb(e)$ is defined as

(5.1) $$\varepsilon(eb(e)) = \frac{d(eb(e))}{de}\frac{e}{eb(e)}.$$

Expanding the differential yields the results obtained in the textbook

(5.2) $$\varepsilon = \frac{d(eb(e))}{de}\frac{e}{eb(e)} = (b(e) + eb'(e))\frac{1}{b(e)} = 1 + \frac{eb'(e)}{b(e)}.$$

Subquestion (b):

The integral of equation (5.5) has the following solution:

(5.3) $$\begin{aligned}&\int_0^{t_1} be^{-bt}e^{-rt}y\,dt = by\int_0^{t_1} e^{-(r+b)t}dt = by\int_0^{t_1} e^{-(r+b)t}dt\\ &= -\frac{by}{r+b}\int_0^{t_1} de^{-(r+b)t} = -\frac{by}{r+b}e^{-(r+b)t}\Big|_{t=0}^{t=t_1}\\ &= -\frac{by}{r+b}\left(e^{-(r+b)t_1} - e^{-(r+b)\times 0}\right) = \frac{by}{r+b}\left(1 - e^{-(r+b)\Delta t}\right),\end{aligned}$$

since $\Delta t = t_1$. Thus, (5.5) in the text becomes

(5.4) $$\begin{aligned}&E = \frac{by}{r+b}\left(1 - e^{-(r+b)\Delta t}\right) + e^{-r\Delta t}\left(e^{-r\Delta t}E + \left(1 - e^{-b\Delta t}\right)U\right)\\ &\Leftrightarrow E\left(1 - e^{-(r+b)\Delta t}\right) = \frac{by}{r+b}\left(1 - e^{-(r+b)\Delta t}\right) + e^{-r\Delta t}\left(1 - e^{-b\Delta t}\right)U.\\ &\Leftrightarrow E = \frac{by}{r+b} + \frac{e^{-r\Delta t}\left(1 - e^{-b\Delta t}\right)}{1 - e^{-(r+b)\Delta t}}U,\end{aligned}$$

which is (5.6) in the text.

Subquestion (c):

L´Hospital´s Rule provides a method for evaluating the limit $\lambda \to 0$. According to L´Hospital´s Rule

(5.5) $$\lim_{\lambda \to 0} \frac{f(\lambda)}{g(\lambda)} = \lim_{\lambda \to 0} \frac{f'(\lambda)}{g'(\lambda)} .$$

Taking the first derivatives of the numerator and the denominator of $\lim_{\Delta t \to 0} \frac{e^{-r\Delta t}\left(1-e^{-b\Delta t}\right)}{1-e^{-(r+b)\Delta t}}$ yields:

(5.6) $$f'(\Delta t) = -re^{-r\Delta t}\left(1-e^{-b\Delta t}\right) + be^{-r\Delta t}e^{-b\Delta t} ,$$

(5.7) $$g'(\Delta t) = (r+b)e^{-(r+b)\Delta t} .$$

Therefore, we have

(5.8) $$\begin{aligned} \lim_{\Delta t \to 0} \frac{e^{-r\Delta t}\left(1-e^{-b\Delta t}\right)}{1-e^{-(r+b)\Delta t}} &= \lim_{\Delta t \to 0} \frac{f(\Delta t)}{g(\Delta t)} = \lim_{\Delta t \to 0} \frac{f'(\Delta t)}{g'(\Delta t)} \\ &= \lim_{\Delta t \to 0} \frac{-re^{-r\Delta t}\left(1-e^{-b\Delta t}\right) + be^{-r\Delta t}e^{-b\Delta t}}{(r+b)e^{-(r+b)\Delta t}} = \frac{b}{r+b}. \end{aligned}$$

Exercise 5.2: Hazard Function (Footnote 39, p. 171)

(a) Exemplify the definition of the hazard function $h(t)$.
(b) Derive $h(t)$ for the exponential distribution and the weibull distribution.

Solutions:

Subquestion (a): The hazard rate is defined as the probability per time unit that a case which has survived until the beginning of the respective interval will fail within that interval. Specifically, it is computed as the number of failures per time units in the respective interval, divided by the average number of surviving cases at the mid-point of the interval. The hazard function can be expressed as the ratio of the probability density function $f(t)$ to the survival function $S(t)$, i.e.

(5.9) $$h(t) = \frac{f(t)}{S(t)} = \frac{f(t)}{\int_t^\infty f(u)du} = \frac{f(t)}{1-F(t)} ,$$

where $F(t)$ is the cumulative distribution function. The relationship between the survival and hazard function is given by

(5.10) $$S(t) = e^{-\int_0^t h(u)du}$$

Subquestion (b): For the exponential distribution we obtain

(5.11) $$\begin{aligned} f(t) &= \lambda e^{-\lambda t} \\ S(t) &= e^{-\lambda t} \\ h(t) &= \lambda \end{aligned}$$

The important feature to note is that the hazard rate is constant, i.e. it does not depend upon time. The weibull distribution is a generalisation of the exponential distribution and has two parameters, λ and γ. λ is usually referred to as the scale parameter, while γ is referred to as the shape parameter.

(5.12) $$f(t)=\lambda\gamma^{\gamma-1}e^{-\lambda t^{\gamma}}$$
$$S(t)=e^{-\lambda t^{\gamma}}$$
$$h(t)=\lambda\gamma t^{\gamma-1}$$

Exercise 5.3:

Derive equation (5.33) on p. 193 of the textbook.

Solution:

By moving the term $(s+q(\theta(t)))(J(t)-V(t))$ of equation (5.32) in the textbook to the left-hand side, we have,

(5.13) $$(r+s+q(\theta(t)))(J(t)-V(t))=(y-w(t)+c)+(\dot{J}(t)-\dot{V}(t)).$$

Let $U(t)=(J(t)-V(t))$, we have

(5.14) $$(r+s+q(\theta(t)))U(t)=(y-w(t)+c)+\frac{dU(t)}{dt}.$$

We can interpret $r+s+q(\theta(t))$ as an effective discount rate at t, and $y-w(t)+c$ the immediate payoff at t. Therefore, as per the discussion of the Bellman equation in Chapter 2, $U(t)$ can be characterised as the following intertemporal value function with the following form:

(5.15) $$U(t_0)=\int_{t_0}^{\infty}(y-w(t)+c)e^{-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau}dt.$$

Alternatively, we can derive the above equation through a more indirect route. Multiplying both sides of (5.14) by $e^{-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau}$ yields

(5.16) $$(r+s+q(\theta(t)))U(t)e^{-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau}dt$$
$$=(y-w(t)+c)e^{-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau}dt+e^{-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau}dU(t).$$

It is easy to see that

(5.17) $$d\left(U(t)e^{-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau}\right)$$
$$=d\left(-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau\right)U(t)e^{-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau}+e^{-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau}dU(t)$$
$$=-(r+s+q(\theta(t)))U(t)e^{-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau}dt+e^{-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau}dU(t).$$

Thus, (5.16) becomes

(5.18) $$-d\left(U(t)e^{-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau}\right)=\left(y-w(t)+c\right)e^{-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau}dt.$$

Integrating both sides of the above equation from t_0 to infinity gives

(5.19) $$\begin{aligned}&-\int_{t_0}^{\infty}d\left(U(t)e^{-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau}\right)dt=\int_{t_0}^{\infty}\left(y-w(t)+c\right)e^{-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau}dt\\ &\Leftrightarrow -U(t)e^{-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau}\Big|_{t=t_0}^{t=\infty}=\int_{t_0}^{\infty}\left(y-w(t)+c\right)e^{-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau}dt\\ &\Leftrightarrow -U(\infty)e^{-\int_{t_0}^{\infty}[r+s+q(\theta(\tau))]d\tau}+U(t_0)e^{-\int_{t_0}^{t_0}[r+s+q(\theta(\tau))]d\tau}=\int_{t_0}^{\infty}\left(y-w(t)+c\right)e^{-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau}dt\end{aligned}$$

With the transversality conditions $U(\infty)e^{-\int_{t_0}^{\infty}[r+s+q(\theta(\tau))]d\tau}=0$ and $U(t_0)=U(t_0)e^{-\int_{t_0}^{t_0}[r+s+q(\theta(\tau))]d\tau}$, we have

(5.20) $$U(t_0)=\int_{t_0}^{\infty}\left(y-w(t)+c\right)e^{-\int_{t_0}^{t}[r+s+q(\theta(\tau))]d\tau}dt\,.$$

Substituting $U(t)=\left(J(t)-V(t)\right)$ back into the above equation gives us (5.33) of the textbook.

Exercise 5.4:

Derive equations (5.55) – (5.57) on p. 201 of the textbook.

<u>Solution:</u>

Substituting $q(\theta)=p(\theta)/\theta$ into equation (5.48) of the textbook gives

(5.21) $$J(t)=\frac{c\theta}{p(\theta)}\equiv\frac{c\theta(t)}{p(\theta(t))}.$$

Differentiating $J(t)$ with respect to *t* gives

(5.22) $$\dot{J}(t)=\frac{c\dot{\theta}(t)}{p(\theta(t))}-\frac{c\theta(t)}{p(\theta(t))^2}p'(\theta(t))\dot{\theta}(t)\Leftrightarrow \dot{J}(t)=\frac{c}{p(\theta(t))}\left[1-\frac{\theta(t)p'(\theta(t))}{p(\theta(t))}\right]\dot{\theta}(t)..$$

The definition of the elasticity of $p(\theta)$ with respect to θ is denoted by

(5.23) $$\eta(p(\theta))=\frac{dp(\theta)}{d\theta}\frac{\theta(t)}{p(\theta(t))}\equiv\frac{\theta(t)p'(\theta(t))}{p(\theta(t))}.$$

Substituting into (5.22) yields

(5.24) $$\dot{J}(t)=\frac{c}{p(\theta(t))}\left[1-\eta(\theta(t))\right]\dot{\theta}(t).,$$

which is equation (5.55) of the textbook.

Substituting (5.49) of the textbook, $\dot{J}(t)=(r+s)J(t)-(y-w(t))$, and $J(t)=\dfrac{c\theta(t)}{p(\theta(t))}$ into the above equation yields equation (5.56) of the textbook,

$$\text{(5.25)} \qquad \frac{c}{p(\theta(t))}(1-\eta)\dot{\theta}(t)=(r+s)\frac{c\theta(t)}{p(\theta(t))}-(y-w(t)).$$

Moving the term $\dfrac{c}{p(\theta(t))}(1-\eta)$ to the right-hand side gives

$$\text{(5.26)} \qquad \dot{\theta}(t)=\frac{r+s}{1-\eta}\theta(t)-\frac{p(\theta(t))}{c(1-\eta)}(y-w(t)).$$

By substituting (5.54) of the textbook, $w(t)=z+\beta(y+c\theta(t)-z)$, into the above, one can obtain equation (5.77) of the textbook:

$$\text{(5.26)} \qquad \begin{aligned} &\dot{\theta}(t)=\frac{r+s}{1-\eta}\theta(t)-\frac{p(\theta(t))}{c(1-\eta)}(y-(z+\beta(y+c\theta(t)-z))) \\ \Leftrightarrow\ &\dot{\theta}(t)=\frac{r+s}{1-\eta}\theta(t)-\frac{p(\theta(t))}{c(1-\eta)}[(1-\beta)(y-z)-\beta c\theta(t)]. \end{aligned}$$

Exercise 5.5:

Derive equations (5.69) – (5.70) on p. 209 of the textbook.

<u>Solution:</u>

For the maximisation problem, (5.67) of the textbook, subject to (5.68) of the textbook:

$$\max_X \int_0^\infty [F(N)-zN-cX]e^{-rt}dt \text{, s.t. } \frac{dN}{dt}=q\left(\frac{X}{L-N}\right)X-sN \text{, } \theta=\frac{X}{L-N} \text{,}$$

the Hamiltonian is denoted by

$$\text{(5.27)} \qquad H=[F(N)-zN-cX]e^{-rt}+\lambda_t\left(q\left(\frac{X}{L-N}\right)X-sN\right)$$

or

$$\text{(5.28)} \qquad H=\left[F(N)-zN-cX+\mu_t\left(q\left(\frac{X}{L-N}\right)X-sN\right)\right]e^{-rt} \text{,}$$

where $\lambda_t=\mu_t e^{-rt}$ represents the shadow price with respect to state variable, *N*.

The first FOC of equation (5.28) with respect to *X* is denoted by

$$\frac{\partial H}{\partial X} = \left[-c + \mu_t \left(q'\left(\frac{X}{L-N} \right) \frac{X}{L-N} + q\left(\frac{X}{L-N} \right) \right) \right] e^{-rt} = 0$$

(5.29)
$$\Leftrightarrow \mu_t \left(q'\left(\frac{X}{L-N} \right) \frac{X}{L-N} + q\left(\frac{X}{L-N} \right) \right) = c$$
$$\Leftrightarrow \mu_t \left(q'(\theta)\theta + q(\theta) \right) = c$$
$$\Leftrightarrow \mu_t = \frac{c}{q'(\theta)\theta + q(\theta)}.$$
,

The second FOC of equation (5.28) with respect to *N* is represented by

$$\frac{\partial H}{\partial N} = -\frac{d\lambda_t}{dt} = -\frac{d\left(\mu_t e^{-rt}\right)}{dt} = -e^{-rt}\dot{\mu}_t + re^{-rt}\mu_t$$

$$\Leftrightarrow \frac{\partial \left[\left[F(N) - zN - cX + \mu_t \left(q\left(\frac{X}{L-N} \right) X - sN \right) \right] e^{-rt} \right]}{\partial N} = -e^{-rt}\dot{\mu}_t + re^{-rt}\mu_t$$

(5.30)
$$\Leftrightarrow \left(F'(N) - z + \mu_t q'\left(\frac{X}{L-N} \right) \frac{X}{(L-N)^2} X - \mu_t s \right) e^{-rt} = -e^{-rt}\dot{\mu}_t + re^{-rt}\mu_t \quad ,$$
$$\Leftrightarrow F'(N) - z + \mu_t q'(\theta)\theta^2 + \mu_t s = -\dot{\mu}_t + r\mu_t$$
$$\Leftrightarrow F'(N) - z = \left(s + r - q'(\theta)\theta^2 \right)\mu_t - \dot{\mu}_t ,$$

which is equation (5.70) of the textbook.

Exercise 5.6: Strategic Foundations of Coordination Games – A Reminder

In the second generation model of currency crisis, two speculators are deciding whether they will attack a currency or not. The attack is successful if both simultaneously decide to attack. The two players (speculators) are called 1 and 2. Both can attack (strategy *A*) or refrain from doing so (strategy *B*). If the players refrain from attacking, their payoff is 0. If both players attack collectively, they both get the payoff *p*. If only one player attacks, then the attack fails and that player receives a payoff of *p*-1.

Figure 5.1: The Payoffs of the One-Shot Game

		Player 2: A	Player 2: B
Player 1	A	p, p	p-1, 0
	B	0, p-1	0, 0

(a) Determine the dominant strategies for (*i*) $p > 1$, (*ii*) $p < 0$, and (*iii*) $0 < p < 1$.
(b) Determine the Nash equilibria for (*i*) $p > 1$, (*ii*) $p < 0$, and (*iii*) $0 < p < 1$.

Solutions:

Subquestion (a):

(*i*) If $p > 1$, *A* is the dominant strategy for both speculators.
(*ii*) If $p < 0$, *B* is the dominant strategy for both speculators.

(*iii*) If $0 < p < 1$, there are two dominant strategies in the game.

Subquestion (b):

(*i*) If $p > 1$, (*A*,*A*) is the dominant strategy for both speculators.
(*ii*) If $p < 0$, (*B*,*B*) is the dominant strategy for both speculators.
(*iii*) If $0 < p < 1$, (*A*,*A*) and (*B*,*B*) are Nash equilibria and thus a coordination game exists in which the optimal strategy depends upon expectations. Both speculators would like to coordinate to attain the equilibrium (*A*,*A*) and sudden and exogenous shifts in expectations may trigger a crisis.

Exercise 5.7: Expectation Traps in Obstfeld´s (1994, 1996) Second Generation Model of Currency Crisis

The ingredients of the second generations model of currency crisis are as follows: The government is minimising the loss function

(5.31) $$L = \left\{(y - y^*)^2 + \beta\pi^2 + \eta R\right\},$$

subject to the expectations-augmented Phillips curve

(5.32) $$y = \bar{y} + \alpha\left(\pi - \pi^e\right) - \varepsilon$$

where y is output, y^* is the output target of the government, $\bar{y}$ is natural output, π is the rate of devaluation, π^e is the expected rate of devaluation, ε is a random supply shock, and R is an indicator that takes the value of 1 if $\pi \neq 0$ and 0 if $\pi = 0$. The parameter η measures the reputation loss of abandoning the fixed exchange rate regime. The model assumes that the government is using the fixed exchange rate regime as a nominal anchor. The reason is that in this stylised fixed exchange rate setting, the fact that there is no inflation abroad means that there is no home inflation either. Viewed from a different perspective, the framework implies that the cost of defending the peg increases with expectations of devaluation.
The benefits of maintaining the peg are lower volatility, lower inflation and enhanced reputation. On the other hand, the costs of maintaining the peg are higher interest rates and lower output. The timing of the game is such that private agents move first, setting π^e without knowing ε. The government moves last setting π after observing ε and knowing π^e.

(a) Determine the welfare loss of the government for keeping the peg versus abandoning the peg.
(b) Determine the equilibrium for alternative ε´*s* and η´s and illustrate the result graphically.

Solution:

Subquestion (a):

We first derive the best response $f\left(\pi^e \varepsilon\right)$ of the government. With $y = \bar{y} + \alpha\left(\pi - \pi^e\right) - \varepsilon$ we have

(5.33) $$\begin{aligned} L &= (y - y^*)^2 + \beta\pi^2 + \eta R \\ &= \left(\bar{y} + \alpha\left(\pi - \pi^e\right) - \varepsilon - y^*\right)^2 + \beta\pi^2 + \eta R \end{aligned}$$

The FOC with respect to π is then denoted by

$$
\begin{aligned}
&\frac{dL}{d\pi} = 2\alpha\left(\bar{y} + \alpha\left(\pi - \pi^e\right) - \varepsilon - y^*\right) + 2\beta\pi = 0 \\
&\Leftrightarrow \frac{dL}{d\pi} = 2\alpha\left(\bar{y} - \alpha\pi^e - \varepsilon - y^*\right) + 2\alpha^2\pi + 2\beta\pi = 0 \\
&\Leftrightarrow 2\left(\alpha^2 + \beta\right)\pi = 2\alpha\left(y^* - \bar{y} + \alpha\pi^e + \varepsilon\right) \\
&\Leftrightarrow \pi = \frac{\alpha\left(y^* - \bar{y} + \varepsilon + \alpha\pi^e\right)}{\alpha^2 + \beta}.
\end{aligned} \tag{5.34}
$$

Substituting the optimal π into the loss function, we have

$$
\begin{aligned}
L_{flex} &= \left(y - y^*\right)^2 + \beta\left(\frac{\alpha\left(y^* - \bar{y} + \varepsilon + \alpha\pi^e\right)}{\alpha^2 + \beta}\right)^2 + \eta \\
&= \left(\bar{y} + \alpha\left(\frac{\alpha\left(y^* - \bar{y} + \varepsilon + \alpha\pi^e\right)}{\alpha^2 + \beta} - \pi^e\right) - \varepsilon - y^*\right)^2 + \beta\left(\frac{\alpha\left(y^* - \bar{y} + \varepsilon + \alpha\pi^e\right)}{\alpha^2 + \beta}\right)^2 + \eta \\
&= \left(\frac{-\beta\left(y^* - \bar{y} + \varepsilon + \alpha\pi^e\right)}{\alpha^2 + \beta}\right)^2 + \beta\left(\frac{\alpha\left(y^* - \bar{y} + \alpha\pi^e\right)}{\alpha^2 + \beta}\right)^2 + \eta \\
&= \frac{\beta\left(\alpha^2 + \beta\right)}{\left(\alpha^2 + \beta\right)^2}\left(y^* - \bar{y} + \varepsilon + \alpha\pi^e\right)^2 + \eta \\
&= \frac{\beta}{\left(\alpha^2 + \beta\right)}\left(y^* - \bar{y} + \varepsilon + \alpha\pi^e\right)^2 + \eta.
\end{aligned} \tag{5.35}
$$

If, instead the government maintains the peg and sets $\pi = 0$, then the losses are

$$
L_{peg} = \left(y^* - \bar{y} + \varepsilon + \alpha\pi^e\right)^2. \tag{5.36}
$$

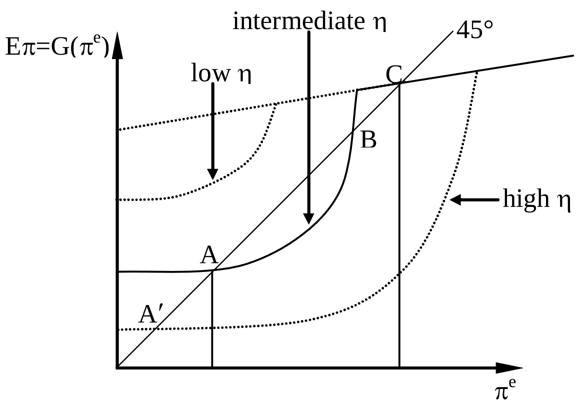
<u>Subquestion (b):</u>

In order to determine the shocks that trigger multiple equilibria, define $\breve{\varepsilon} = \breve{\varepsilon}\left(\pi^e\right)$ and $\hat{\varepsilon} = \hat{\varepsilon}\left(\pi^e\right)$ as the lowest and highest solution to $L_{flex} = L_{peg}$. Whenever $\varepsilon \in \left(\breve{\varepsilon}, \hat{\varepsilon}\right)$, the government finds it optimal to maintain the peg and set $\pi = 0$. Whenever $\varepsilon \notin \left(\breve{\varepsilon}, \hat{\varepsilon}\right)$, the government prefers to allow the exchange rate to float. In equilibrium

$$
\pi^e = 0 \cdot \text{prob}\left(\varepsilon \in \left(\breve{\varepsilon}, \hat{\varepsilon}\right)\right) + f\left(\pi^e, \varepsilon\right)\text{prob}\left(\varepsilon \notin \left(\breve{\varepsilon}, \hat{\varepsilon}\right)\right) \equiv G\left(\pi^e\right) \tag{5.37}
$$

It can be shown that $G(0) > 0$, $G' > 0$, and over some range, $G' > 1$. The implication is that, depending upon η, the model either has a unique equilibrium or multiple fixed points. The graph below illustrates this.

Figure 5.2: Multiple Equilibria in the Obstfeld (1994) Model



For either very small or very large values of η, the resulting equilibrium is unique. For intermediate values of η, however, multiple equilibria occur which are represented by *A*, *B* and *C*. In the "bad" equilibrium, *C*, the peg is always abandoned. On the contrary, in the "good" equilibrium, *A*, the peg is only abandoned for extreme shocks. Note the circular logic: if everybody is expecting *C*, then it is optimal to attack. On the contrary, if everyone expects *A*, then not attacking is the optimal choice. Thus, there are self-fulfilling crises in the modelling framework. So-called "sunspots", which may be completely unrelated to the economy, may change expectations and trigger a currency crisis.

Additional Reference:

Obstfeld, M. (1994) "The Logic of Currency Crisis", *Cahier Economiques et Monétaires* 43, 189-213.

Obstfeld, M. (1996) "Models of Currency Crisis with Self-Fulfilling Features", *European Economic Review* 40, 1037-1047.

Exercise 5.8: Investment Complementarities (pp. 211-216):

Consider a modelling set-up where the payoff for agent *i* is given by $v_i = V(e_i, E, \theta) = A(E, \theta)e_i - c(e_i)$, where e_i is the effort level which can be interpreted as investment, *E* is aggregate investment, $A(\cdot)$ are the gross returns to investment, $c(\cdot)$ are the cost of investment, and θ is an exogenous productivity shock. We assume $V_\theta > 0$, $A_\theta > 0$ and $c' > 0$. Furthermore, we assume that an investment complementarity exists. This is equivalent to assume $A_E > 0$ and $V_{eE} > 0$.

(a) In order to keep the model tractable, suppose that $c(\cdot) = e_i^2/2$. Deduce the best response function and the symmetric market equilibrium and illustrate the obtained solution graphically. Present the graph in the (e_i,E) space and introduce the concepts of weak and strong complementarity.
(b) Prove that strong complementarity delivers amplification and co-movements.

Solution:

Subquestion (a): The best response function is given as

(5.38) $$e_i = \arg\max_e V(e_i, E, \theta).$$

(5.39) $$\frac{\partial V}{\partial e_i} = A(E, \theta) - \frac{2e_i}{2} = 0 \;\Leftrightarrow\; e_i = A(E, \theta)$$

The optimal investment of agent *i* increases with the aggregate capital stock and the productivity shock. In the symmetric equilibrium all agents choose the same effort (investment) level and thus aggregate investment is solved as $E = A(E, \theta)$, the steepness of the best response function. Two possibilities are illustrated Figures 5.3 and 5.4 below.

Figure 5.3: The Symmetric Equilibria for Weak and Strong Complementarities

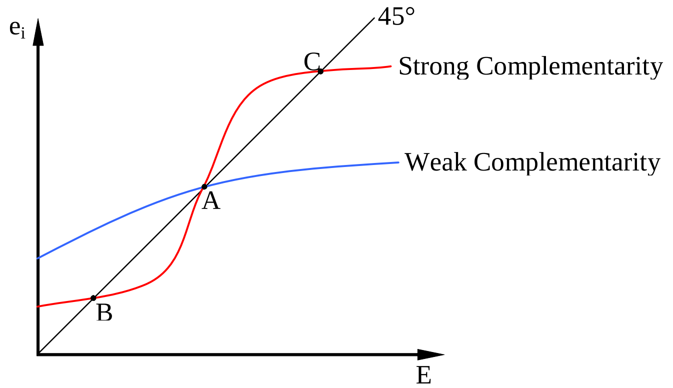
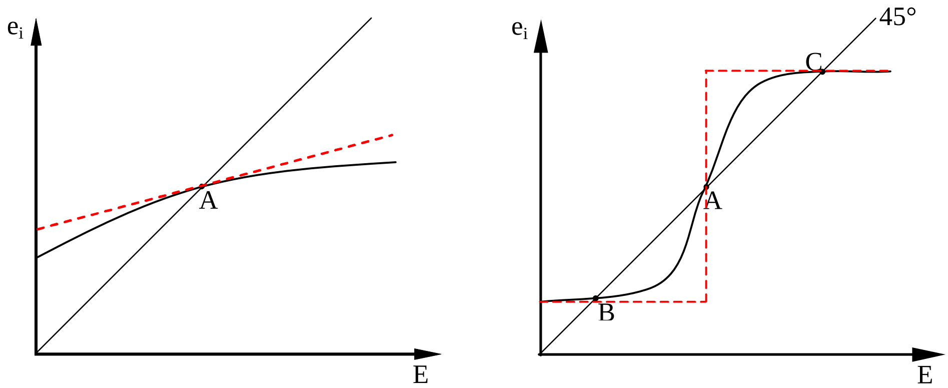


Figure 5.4: Best Responses with Weak and Strong Complementarities



The case of *weak* complementarity corresponds to the case when the best respond function intersects only once with the 45° degree line. Point *A* then gives the unique investment equilibrium in the economy. *Strong* complementarity corresponds to the situation when the best response function intersects three times with the 45° degree line, giving rise to one unstable equilibrium (*A*) and two stable equilibria (*B* and *C*). The corresponding best responses are drawn in Figure 5.4 as dashed lines. The implication is that the curvature of the best response function determines the existence of multiple equilibria in the economy.

<u>Subquestion (b):</u> From the implicit function theorem we have $E = E(\theta)$. Differentiating *E* yields

(5.40) $$\frac{dE}{d\theta} = \frac{A_\theta}{1 - A_E}.$$

Without complementarities, A_E = 0 and therefore $\frac{dE}{d\theta} = A_\theta$. Now suppose complementarities exist. In any stable equilibrium $A_E < 1$. Therefore, strong complementarities amplify shocks and the multiplier is $\frac{1}{1-A_E} > 1$.

Suppose that productivity is idiosyncratic, i.e. $e_i = A(E, \theta_i)$ where θ_i is distributed with density $f(\cdot)$. An equilibrium $E = E(\theta)$ is now

(5.41) $\quad E = E(\theta) = A\left(\int E(z) f(z) dz, \theta\right)$.

In the absence of complementarities, the effort/investment of *i* depends only on θ_i. In contrast, in the case of strong complementarities, the investment of *i* also depends upon other agents´ productivities leading to co-movements in the economy.

Exercise 5.8: Strategic Complementarities between Human Capital and R&D

In the paper "The Low-Skill, Low-Quality Trap: Strategic Complementarities Between Human Capital and R&D" (*The Economic Journal* 106, 1996, 458-470), Stephen Redding models the complementarities between human capital and R&D in an endogenous growth model. Read the paper carefully and conduct the following exercises:

(a) Derive equation (11) on p. 462.
(b) Derive equation (15) on p. 464.
(c) Derive equation (16) on p. 464.
(d) Deduce proposition 3 on p. 467.
(e) Deduce proposition 4 on p. 467.

Solution:

Subquestion (a): The maximization of equation (10) of Redding (1996) is represented by

(5.42) $$\max_v \beta\left[(1-v) + \left(\frac{1}{1+\rho}\right)[(\mu\lambda + (1-\mu))](1+\gamma v^\theta)\right]A_{1,m}(1-\delta)H_{2,t-1},$$

where v is the control variable for individuals deciding in period 1 the fraction of time to spend on schooling or human capital accumulation and $0 \le v \le 1$, ρ is time-preference discount rate, μ is the Poisson probability of innovation, $\lambda > 1$ denotes innovation, $\gamma > 0$ and $0 < \theta < 1$ are the parameters scaling the productivity of the education technology, δ is the human capital depreciation rate, β represents the fraction of surplus workers, $A_{1,m}$ captures the productivity of the technology employed in period 1 with m number of innovations that have occurred, and finally $H_{2,t-1}$ is the aggregate period 2 stock of human capital of generation *t* -1. The first order condition of the maximization problem presented above is denoted by:

(5.43) $$\beta\left[-1 + \left(\frac{1}{1+\rho}\right)[\mu\lambda + (1-\mu)]\gamma\theta v^{\theta-1}\right]A_{1,m}(1-\delta)H_{2,t-1} = 0$$

The term in bracket gives

$$\left(\frac{1}{1+\rho}\right)[\mu\lambda+(1-\mu)]\gamma\theta v^{\theta-1}=1$$

(5.44) $$\Leftrightarrow v^{1-\theta}=\frac{\gamma\theta[\mu\lambda+(1-\mu)]}{1+\rho}$$

$$\Leftrightarrow v=\left[\frac{\gamma\theta[\mu\lambda+(1-\mu)]}{1+\rho}\right]^{\frac{1}{1-\theta}}.$$

The variable v must conform to the constraint of $0\le v\le 1$. Therefore, the first order condition becomes

(5.45) $$v=\begin{cases}\left[\frac{\gamma\theta[\mu\lambda+(1-\mu)]}{1+\rho}\right]^{\frac{1}{1-\theta}} & \text{for } 0\le\left[\frac{\gamma\theta[\mu\lambda+(1-\mu)]}{1+\rho}\right]^{\frac{1}{1-\theta}}\le 1\\ 1 & \text{for } \left[\frac{\gamma\theta[\mu\lambda+(1-\mu)]}{1+\rho}\right]^{\frac{1}{1-\theta}}>1.\end{cases}$$

which is equation (11) of Redding (1996).

Subquestion (b): For the optimal research effort in the high growth equilibrium, it is required that the following relation holds

(5.46) $$V(R)>V(0),$$

where

(5.47) $$V(R)=(1-\beta)\left[(1-\alpha')(1-v)+\left(\frac{1}{1+\rho}\right)[\psi\lambda+(1-\psi)](1+\gamma v^{\theta})\right]A_{1,m}h_{1,t}$$

and

(5.48) $$V(0)=(1-\beta)\left[(1-v)A_{1,m}h_{1,t}+\left(\frac{1}{1+\rho}\right)A_{1,m}(1+\gamma v^{\theta})h_{1,t}\right].$$

With $v=v_{\psi}$, we then have

$$V(R)>V(0)$$

(5.49) $$\Leftrightarrow (1-\beta)\left[(1-\alpha')(1-v_{\psi})+\left(\frac{1}{1+\rho}\right)[\psi\lambda+(1-\psi)](1+\gamma v_{\psi}^{\theta})\right]A_{1,m}h_{1,t}$$

$$>(1-\beta)\left[(1-v_{\psi})A_{1,m}h_{1,t}+\left(\frac{1}{1+\rho}\right)A_{1,m}(1+\gamma v_{\psi}^{\theta})h_{1,t}\right]$$

Removing the common factor $(1-\beta)A_{1,m}h_{1,t}$ from the above relationship gives

(5.50) $$(1-\alpha')(1-v_{\psi})+\left(\frac{1}{1+\rho}\right)[\psi\lambda+(1-\psi)](1+\gamma v_{\psi}^{\theta})>(1-v_{\psi})+\left(\frac{1}{1+\rho}\right)(1+\gamma v_{\psi}^{\theta}).$$

Collecting terms finally yields

(5.51)
$$-\alpha'(1-v_\psi)+\left(\frac{1}{1+\rho}\right)\psi(\lambda-1)(1+\gamma v_\psi^\theta)>0$$
$$\Leftrightarrow \frac{\psi(\lambda-1)}{1+\rho}>\frac{\alpha'(1-v_\psi)}{1+\gamma v_\psi^\theta},$$

which is equation (15) of Redding (1996).

Subquestion (c): For the low growth equilibrium with $v = v_0$ and $\mu = 0$, we have

(5.52)
$$V(R)<V(0)$$
$$\Leftrightarrow (1-\beta)\left[(1-\alpha')(1-v_0)+\left(\frac{1}{1+\rho}\right)[\psi\lambda+(1-\psi)](1+\gamma v_0^\theta)\right]A_{1,m}h_{1,t}$$
$$>(1-\beta)\left[(1-v_0)A_{1,m}h_{1,t}+\left(\frac{1}{1+\rho}\right)A_{1,m}(1+\gamma v_0^\theta)h_{1,t}\right]$$

Following the same procedure as in (b), we have

(5.53)
$$\frac{\psi(\lambda-1)}{1+\rho}<\frac{\alpha'(1-v_0)}{1+\gamma v_0^\theta},$$

which is equation (16) of Redding (1996).

Subquestion (d): The expected rate of growth of final goods output is denoted by [see Redding (1996), p. 466]

(5.54)
$$\log\left(\frac{E_{m,i}Y_{t+1}}{Y_t}\right)=\log\left(E_{m,i}\int_0^1 A_{1,t+1}(i)di\right)-\log\left(\int_0^1 A_{1,t}(i)di\right)$$
$$+\log(E_{m,i}H_{1,t+1})-\log(H_{1,t}).$$

With $E_{m,i}H_{1,t+1}=(1-\delta)H_{1,t}E_{m,i}\int_0^1\left[1+\gamma v(i)^\theta\right]di$, $\log\left(E_{m,i}\int_0^1 A_{1,t+1}(i)di\right) = \log\{[\mu\lambda+(1-\mu)]A_{1,t}\}$, and $\int_0^1 A_{1,t}(i)di = A_{1,t}$ we have

(5.55)
$$\log\left(\frac{E_{m,i}Y_{t+1}}{Y_t}\right)=\log[\mu\lambda+(1-\mu)]+\log(A_{1,t})-\log(A_{1,t})$$
$$+\log\left((1-\delta)E_{m,i}\int_0^1\left[1+\gamma v(i)^\theta\right]di\right)+\log(H_{1,t})-\log(H_{1,t}).$$
$$\Leftrightarrow \log\left(\frac{E_{m,i}Y_{t+1}}{Y_t}\right)=\log[\mu\lambda+(1-\mu)]+\log\left((1-\delta)\int_0^1\left[1+\gamma v(i)^\theta\right]di\right).$$

For a "high growth" equilibrium so that $v = v_\psi$, we have

(5.56)
$$\log\left(\frac{E_{m,i}Y_{t+1}}{Y_t}\right)=\log[\mu\lambda+(1-\mu)]+\log\left((1-\delta)\int_0^1\left[1+\gamma v_\psi^\theta\right]di\right).$$

For a “low growth” equilibrium so that $v = v_0$ and $\mu = 0$

$$\text{(5.57)} \qquad \log\left(\frac{E_{m,i}Y_{t+1}}{Y_t}\right) = \log\left((1-\delta)\int_0^1 \left[1+\gamma v_0^{\theta}\right]di\right).$$

From the two equations given above, proposition 3 can be deduced.

<u>Subquestion (e):</u> Proposition 4 relates to the growth rate. Therefore we need to solve the integral $\int_0^1 \left[1+\gamma v^{\theta}\right]di$. It is assumed that *v*, the optimal fraction of time invested in schooling, is constant after optimization, and thus not a function of *i*. After substituting equation (12) from p. 463, the two equations of Proposition 3 can thus be transformed as follows:

For the “high growth” equilibrium we obtain

$$\text{(5.58)} \qquad \begin{aligned} \log\left(\frac{E_{m,i}Y_{t+1}}{Y_t}\right) &= \log\left[\mu\lambda + (1-\mu)\right] + \log\left((1-\delta)\left[1+\gamma v_{\psi}^{\theta}\right]\right) \\ &= \log\left[\mu\lambda + (1-\mu)\right] + \log\left((1-\delta)\left[1+\gamma\left[\frac{\gamma\theta\left[\psi\lambda + (1-\psi)\right]}{1+\rho}\right]^{\frac{\theta}{1-\theta}}\right]\right). \end{aligned}$$

The corresponding growth rate for the “low growth” equilibrium is

$$\text{(5.59)} \qquad \log\left(\frac{E_{m,i}Y_{t+1}}{Y_t}\right) = \log\left((1-\delta)\left[1+\gamma\left[\frac{\gamma\theta}{1+\rho}\right]^{\frac{\theta}{1-\theta}}\right]\right).$$