# Chapter 4: Growth in Dynamic General Equilibrium

## I. Motivational Questions and Exercises:

**Exercise 4.1 (p. 138):**

Consider the limit $\lambda \to 0$ of the *CES* production function $Y = \left[\alpha K^{\lambda} + (1-\alpha) L^{\lambda}\right]^{1/\lambda}$ with $\lambda \leq 1$. Demonstrate that for $\lambda \to 0$ the *CES* production function degenerates to the *CD* production function.

<u>Solution:</u> The *CES* production function can be rewritten as

(4.1) $$\ln Y = \frac{\ln\left[\alpha K^{\lambda} + (1-\alpha) L^{\lambda}\right]}{\lambda}.$$

L´Hospital´s Rule provides a method for evaluating the limit $\lambda \to 0$. According to L´Hospital´s Rule

(4.2) $$\lim_{\lambda \to 0} \frac{f(\lambda)}{g(\lambda)} = \lim_{\lambda \to 0} \frac{f'(\lambda)}{g'(\lambda)}.$$

Taking the first derivatives of the numerator and the denominator of (4.2) yield:

(4.3) $$f'(\lambda) = \frac{\alpha K^{\lambda} \ln K + (1-\alpha) L^{\lambda} \ln L}{\alpha K^{\lambda} + (1-\alpha) L^{\lambda}}$$

and

(4.4) $$g'(\lambda) = 1.$$

Therefore, for $\lambda$ approaching 0, we get:

(4.5) $$\lim_{\lambda \to 0} \frac{f'(\lambda)}{g'(\lambda)} = \frac{\dfrac{\alpha K^{0} \ln K + (1-\alpha) L^{0} \ln L}{\alpha K^{0} + (1-\alpha) L^{0}}}{1} = \alpha \ln K + (1-\alpha) \ln L$$

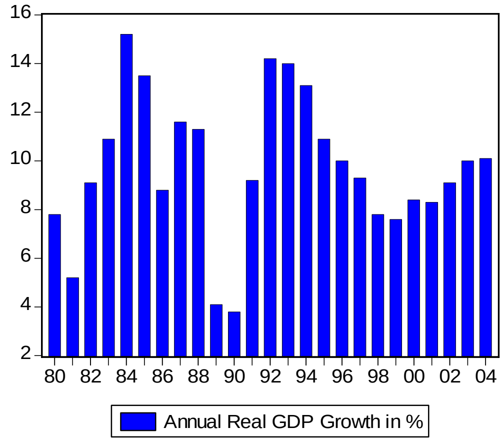
Taking the antilog finally yields the *CD* production function

(4.6) $$Y = K^{\alpha} L^{1-\alpha}.$$

**Exercise 4.2 (Exercise 35 on p. 142 and pp. 252-253):**

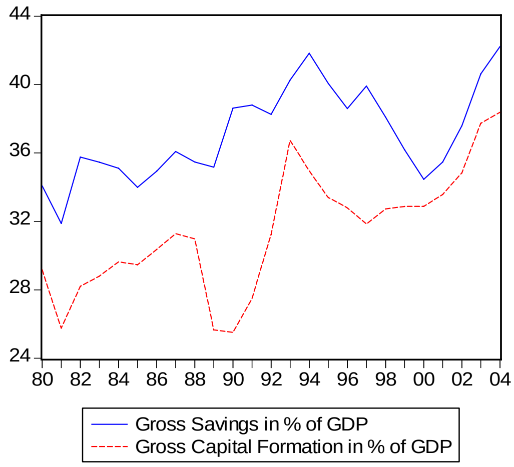
According to the „Golden Rule“, $f'(k) = \delta + g_A + g_N$ is required to maximise steady state consumption. Use the theoretical “Golden Rule” relationship to guide economic thinking about current economic growth in Mainland China. As shown in Figure 4.1, China has become one of the fastest growing economies in the world, expanding at a pace of up to 10 percent per year in recent times (all data are taken from the Worldbank CD-Rom *World Development Indicators 2006*, Washington).

**Figure 4.1: Mainland China´s Economic Growth Path, 1980 - 2004**




Investment has been the most important driver of this growth process. Since the beginning of the 1990s, China's gross investment share of GDP has trended upwards and reached values in the forties, a level well above that of other East Asian economies. Rising fixed capital formation has been fuelled by a corresponding rise in the national saving rate.

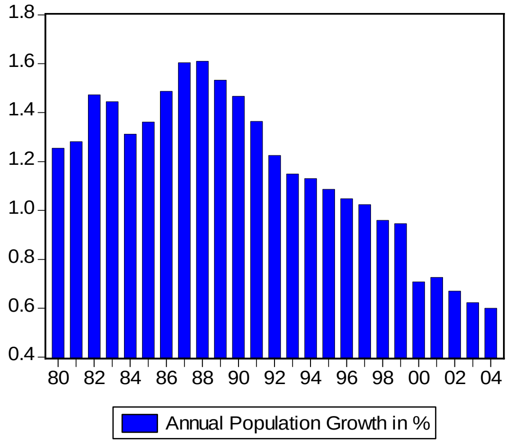
**Figure 4.2: Gross Capital Formation and Gross Savings as Percent of GDP in Mainland China**




Is the share of gross capital formation in the Chinese economy too high, when measured against the "Golden Rule" benchmark? [Hint: In order to answer the question, use the time series for $f'(k)$ provided in Barnett and Brooks (2006, p. 6)]

Solution: In order to decide whether China is at, above, or below the "Golden Rule" steady state benchmark, it is necessary to quantify the marginal product of capital $f'(k)$, $\delta$, $g_A$ and $g_N$. The annual population growth rates are given in Figure 4.3 below. An (upper bound) baseline parameter for the steady state growth rate $g_N$ is 0.00.

**Figure 4.3: Annual Chinese Population Growth Rates, 1980 - 2004**




According to Barnett and Brooks (2006, p. 6), the current marginal productivity of capital $f'(k)$ is approximately 15 percent. A reasonable baseline parameter for depreciation is 0.05 < $\delta$ < 0.07. The long-run steady state growth rate of GDP can be estimated to be in the range 0.02 < $g_A$ < 0.03. Inserting the parameters into the "Golden Rule" $f'(k) = \delta + g_A + g_N$ implies that China is characterised by $f'(k) > \delta + g_A + g_N$, i.e. China has less capital than in the "Golden Rule" steady state. Using the "Golden Rule" as an intellectual foundation, there is therefore no need for China to embark towards a more consumption-driven growth path in order to maximise welfare.


Additional References:

Abel, A.B., Mankiw, N.G., Summers, L.H. and R. Zeckhauser (1989) "Assessing Dynamic Efficiency: Theory and Evidence", *Review of Economic Studies* 56, 1-20.

Barnett, S. and R. Brooks (2006) "What's Driving Investment in China?", *IMF Working Paper WP/06/265*, Washington.


**Exercise 4.3 (p. 143):**

Consider the constant elasticity of marginal utility $-\sigma = u''(c)c/u'(c)$.

(a) Deduce the *CRRA* utility function $u(c) = c^{1-\sigma}/(1-\sigma)$ from the definition of the constant elasticity of marginal substitution given above.

(b) Plot the *CRRA* utility function for $\sigma$ < 1, $\sigma$ = 1, and $\sigma$ > 1, respectively.

Solution:

(a) The elasticity of marginal utility is defined as

(4.7) $$-\sigma = \frac{u''(c)c}{u'(c)}.$$

Let's define $f(c) = u'(c)$. We then obtain the differential equation

(4.8) $$\frac{f'(c)}{f(c)} = \frac{-\sigma}{c} \Leftrightarrow \frac{df}{f} = \frac{-\sigma}{c} dc$$

Integrating (4.8), we get

(4.9) $\ln f = -\sigma \ln c + \ln A$.

Recall that $x = e^{\ln x}$ and $e^{\ln f} = e^{-\sigma \ln c + \ln A}$. We therefore obtain

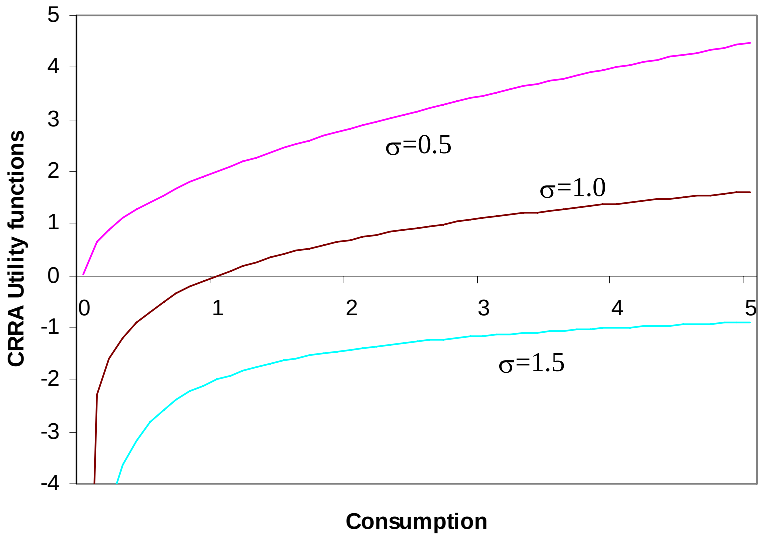
(4.10) $f = c^{-\sigma} A \Leftrightarrow u'(c) = A c^{-\sigma}$.

Finally, integrating (4.10) leads to the *CRRA* utility function

(4.11) $$u(c) = \begin{cases} \frac{A}{1-\sigma} c^{1-\sigma} & \sigma \neq 1 \\ A \ln c & \sigma = 1 \end{cases}.$$

(b)

**Figure 4.4: The *CRRA* Utility Function**




**Exercise 4.4 (pp. 154-156; Learning by Doing)**

Consider an economy with identical firms and technical knowledge embodied in physical capital according to

$$Y = AL^{1-\alpha}K^{\alpha},\ A = (NK)^{\gamma},\ 0 < \alpha < 1 \quad 0 < \gamma < 1$$

where $N$ is the number of firms, $L$ is the level of employment and $K$ is the level of the capital stock, and $A$ denotes knowledge. $A$ is a positive function of the total capital stock $NK$, representing learning by doing in the economy. Determine the behaviour of the economy in a standard Ramsey growth model with *CRRA* utility, keeping in mind the parameters $\alpha$ and $\gamma$.

<u>Solution:</u> The marginal product of capitals can be written as

(4.12) $MPK = \alpha A L^{1-\alpha} K^{\alpha-1} = \alpha A \left(\frac{K}{L}\right)^{-(1-\alpha)} = \alpha A k^{-(1-\alpha)}$,

where $k = K/L$. A consumer in the Ramsey growth model who maximizes intertemporal *CRRA* utility has the following differential equation for consumption,

(4.13) $$\frac{dc}{dt} = \frac{c}{\sigma}\left(f'(k) - \rho\right) = \frac{c}{\sigma}\left(\alpha A k^{-(1-\alpha)} - \rho\right),$$

where $\sigma$ represents constant risk aversion coefficient and $\rho$ is the discount rate of consumers. The relationship between *k* and aggregate capital $\overline{K}$ and aggregate labour $\overline{L}$ is shown by

(4.14) $$k = \frac{K}{L} = \frac{\overline{K}}{\overline{L}},$$

since all firms are identical. Substituting $A = (NK)^{\gamma} = (\overline{K})^{\gamma}$ into the (4. ) yields

(4.15) $$\frac{dc}{dt} = \frac{c}{\sigma}\left(f'(k) - \rho\right) = \frac{c}{\sigma}\left(\alpha\left(\overline{K}\right)^{\gamma}\left(\frac{\overline{K}}{\overline{L}}\right)^{-(1-\alpha)} - \rho\right) = \frac{c}{\sigma}\left(\alpha \overline{L}^{(1-\alpha)}\overline{K}^{\gamma-(1-\alpha)} - \rho\right)$$

It is obvious that three different cases exist.

(a) If $\gamma < 1-\alpha$, *MPK* diminishes over time and approaches $\rho$. The externality of learning by doing is too small to prevent diminishing *MPK*.
(b) If $\gamma > 1-\alpha$, we have increasing *MPK*, and subsequently the growth of consumption explodes.
(c) In the borderline case $\gamma = 1-\alpha$, (4.15) becomes

(4.16) $$\frac{1}{c}\frac{dc}{dt} = \frac{1}{\sigma}\left(\alpha \overline{L}^{1-\alpha} - \rho\right)$$

and the growth rate of consumption is constant over time. The diminishing *MPK* disappears since, for the economy as a whole, the external effect of learning by doing leads to constant returns of scale with respect to *K*:

(4.17) $$Y = NY = NAL^{1-\alpha}K^{\alpha} = N^{\alpha}N^{1-\alpha}(NK)^{\gamma}L^{1-\alpha}K^{\alpha} = \overline{K}^{\gamma}\overline{L}^{1-\alpha}\overline{K}^{\alpha} = \overline{KL}^{1-\alpha},$$

since $\gamma = 1-\alpha$.

**Exercise 4.5 (pp. 157-164; Cross-Country Technology Diffusion in the Romer Model)**

The Romer model describes an economy in which innovative activity requires productive efforts. However, no country uses only technologies that were invented in that country. Rather, products and technologies invented in one country will be used around the world. This exercise builds on the model of Bernard and Jones (1996) to analyse how long-run growth is then determined for individual countries. The key assumption is that total factor productivity works differently for leading economies than for emerging economies. Assume that the leading country has a technology level $A_t$, that grows at the (exogenous) rate *g* according to

$$\frac{\dot{A}_t}{A_t} = g \ .$$

All other countries $i$ have technology levels $B_{it} < A_t$. These technology levels grow according to

$$\dot{B}_{it} = \lambda_i \left( A_t - B_{it} \right),$$

where $\lambda_i$ is a country-specific catching-up and upgrading capability parameter. The differential equation for $B_i$ indicates that, as country $i$ approaches the technological frontier, all the easy gains have already been reaped.

(a) Deduce the steady state solution in which technology in all economies grows at the same rate. Interpret the solution.
(b) Derive the convergence to the steady state.

Solution:

Subquestion (a): The steady state growth rate is given by

$$\begin{aligned} &\frac{\dot{B}_{it}}{B_{it}} = \lambda_i \left( \frac{A_t - B_{it}}{B_{it}} \right) \\ &\Leftrightarrow \lambda_i \left( \frac{A_t - B_{it}}{B_{it}} \right) = g \\ &\Leftrightarrow A_t - B_{it} = \frac{g}{\lambda_i} B_{it} \\ &\Leftrightarrow B_{it} = \frac{\lambda_i}{g + \lambda_i} A_t \ . \end{aligned} \tag{4.18}$$

The solution leads to two conclusions of relevance to follower countries: ($i$) Since the term $\frac{\lambda_i}{g + \lambda_i}$ is smaller than one, the follower country never catches up in the steady state; ($ii$) the higher the learning speed in adopting new technologies $\lambda_i$, the higher the follower country´s long-run level of income will be after the catch-up phase. The message for the catching-up countries, therefore, is encouraging but also contains a cautionary note.

Subquestion (b): Denote $d^2 B_{it} / dt^2 = \ddot{B}_{it}$. Convergence to the steady state is then given by

$$\begin{aligned} &\ddot{B}_{it} = \lambda_i \dot{A}_t - \lambda_i \dot{B}_{it} \\ &\Leftrightarrow \ddot{B}_{it} = g\lambda_i A_t - \lambda_i \dot{B}_{it} \\ &\Leftrightarrow \ddot{B}_{it} = g\lambda_i \left( B_{it} + \frac{\dot{B}_{it}}{\lambda_i} \right) - \lambda_i \dot{B}_{it} \\ &\Leftrightarrow \ddot{B}_{it} + \left( \lambda_i - g \right) \dot{B}_{it} - g\lambda_i B_{it} = 0 \end{aligned} \tag{4.19}$$

The second-order differential equation yields a solution of the form

$$B_t = C_1 e^{\beta_1 t} + C_2 e^{\beta_2 t} \ , \tag{4.20}$$

where $\beta_1$ and $\beta_2$ are the roots of the characteristic equation

(4.21)
$$x^2 + (\lambda_i - g)x - g\lambda_i = 0$$
$$\Leftrightarrow (x - g)(x + \lambda_i) = 0$$

Thus, the general solution is of the form

(4.22)
$$B_t = C_1 e^{gt} + C_2 e^{-\lambda_i t} \ .$$

Bringing (a) and (b) together, a growth trajectory can be mapped. Emerging economies catch up with richer nations, but because the term $e^{-\lambda_i t}$ tends to zero for $t \to \infty$, the growth rates of the follower countries converge towards the (exogenous) growth rate ($g$) of the lead country.

Additional Reference:

Bernard, A. and C. Jones (1996) "Technologiy and Convergence", *Economic Journal* 106, 1037-1044.

**Exercise 4.6 (pp. 161-164; Steady State Growth in the Romer Model)**

In the Romer model output is generated with a two-tier production function. At the lower tier, differentiated intermediate goods are produced (for the microfoundation, see pp. 161-163). At the upper tier, final output production takes the specific form

$$Y = L_Y^{1-\alpha} \sum_{i=1}^{A} x_i^{\alpha} = (AL_Y)^{1-\alpha} X^{\alpha} ,$$

where $Y$ is homogeneous final output, $L_Y$ is the (exogenous) amount of labour input in the final production sector, $x_i$ ($i$ = 1, ..., $A$) are the intermediate goods and $0 < \alpha < 1$. The change in the number of intermediate goods is determined according to

$$\dot{A} = \gamma L_A^{\lambda} A^{\phi} ,$$

where $L_A$ are the *R&D* employees, $0 < \lambda < 1$, and $0 < \phi < 1$. In keeping with the spirit of Solow´s exogenous growth model, we lastly assume

$$\frac{\dot{L}}{L} = n ,$$

$$L_A = s_A L ,$$

$$L = L_A + L_Y ,$$

$$\dot{X} = s_X Y - \delta X ,$$

where $\delta$ is the depreciation rate, $n$ is the growth rate of the labour force, and the $s_A$ and $s_X$ are the savings rates. Since the standard Solow model arises as a special case of the Romer model, the proposed growth model has the advantage of embedding a meaningful benchmark against which to assess the impact of *R&D* on growth.

(a) What is the growth rate along the balanced growth path in the Romer model assuming that all of the intermediate capital goods play an identical role in the production process?
(b) Analyse the convergence dynamics of *A*.
(c) Following Romer´s (1990) original paper, suppose that $\lambda = \phi = 1$. What are the dynamics in this case?

Solution:

Subquestion (a): An identical role of all intermediate goods $i = 1, 2, \ldots, A$ in the production process implies

(4.23) $$x_i = \bar{x} \Leftrightarrow X = A\bar{x}$$

This means that the upper-tier production function can be written as

(4.24) $$\begin{aligned} &Y = AL_Y^{1-\alpha}\,\bar{x}^{\alpha} \\ &\Leftrightarrow Y = \left(AL_Y\right)^{1-\alpha} X^{\alpha} \\ &\Leftrightarrow Y = \left(As_Y L\right)^{1-\alpha} X^{\alpha} \;, \end{aligned}$$

where $s_Y = 1 - s_A$. Taking logs yields

(4.25) $$\left(\frac{\dot{Y}}{Y}\right) = (1-\alpha)\left(\frac{\dot{A}}{A} + \frac{\dot{s}_Y}{s_Y} + \frac{\dot{L}}{L}\right) + \alpha\left(\frac{\dot{X}}{X}\right)$$

In the steady state the growth rates of *Y* and *X* are the same:

(4.26) $$\left(\frac{\dot{Y}}{Y}\right) = (1-\alpha)\left(\frac{\dot{A}}{A} + \frac{\dot{s}_Y}{s_Y} + \frac{\dot{L}}{L}\right) + \alpha\left(\frac{\dot{Y}}{Y}\right).$$

Along the balanced growth path the share of labour allocated to the upper tier production process is constant. Hence we have

(4.27) $$\left(\frac{\dot{Y}}{Y} - \frac{\dot{L}}{L}\right) = \left(\frac{\dot{A}}{A}\right)$$

In the Romer model, the growth rate of *A* is given by

(4.28) $$\begin{aligned} &\frac{\dot{A}}{A} = \gamma L_A^{\lambda} A^{\phi-1} \\ &\Leftrightarrow \frac{\dot{A}}{A} = \gamma \left(s_A L\right)^{\lambda} A^{\phi-1} \end{aligned}$$

Assume that the growth rate of *A* in steady state is constant. Taking logs and derivatives yields

(4.29) $$\lambda\left(\frac{\dot{s}_A}{s_A} + \frac{\dot{L}}{L}\right) - (1-\phi)\left(\frac{\dot{A}}{A}\right) = 0\,.$$

Along the balanced growth path, $s_A = \bar{s}_A$ and therefore $\dot{s}_A / s_A = 0$. The steady state growth rate will then be

(4.30) $$\left(\frac{\dot{Y}}{Y} - \frac{\dot{L}}{L}\right) = \left(\frac{\dot{A}}{A}\right) = \frac{n\lambda}{1-\phi} \ .$$

Intuitively, the steady state growth rate depends upon three factors. (*i*) A higher growth rate of *n* implies that the number of *R&D* employees is increasing [see Kremer (1993)], (*ii*) the parameter $\lambda$ determines to which extent diminishing returns set in as the number of *R&D* employees increases, and finally (*iii*) the parameter $\phi$ determines the extent to which knowledge created in the past makes research today more efficient.

Subquestion (b): The dynamics of *A* over time are given by

(4.31) $$\begin{aligned} & g_A = \frac{\dot{A}}{A} = \gamma L_A^{\lambda} A^{\phi-1} \\ \Leftrightarrow & \frac{\dot{g}_A}{g_A} = \lambda\left(\frac{\dot{s}_A}{s_A} + n\right) - (1-\phi) g_A \ . \end{aligned}$$

Equation (4.31) implies that $g_A$ will be *falling* whenever

(4.32) $$g_A > \frac{\lambda n}{1-\phi} + \frac{\lambda}{1-\phi}\frac{\dot{s}_A}{s_A} \ .$$

Apart from periods when the share of *R&D* employees is changing and therefore $\frac{\dot{s}_A}{s_A} \neq 0$, the growth rate of *A* will be declining whenever it is greater than the steady state value $\frac{n\lambda}{1-\phi}$. Hence, the dynamics of *A* display a tendency towards convergence, always tending back to the balanced growth path.

Subquestion (c): For the special case $\lambda = \phi = 1$ the growth rate of *A* is given by

(4.33) $$\frac{\dot{A}}{A} = \gamma L_A \ ,$$

i.e. the growth rate increases in line with the absolute number of researchers. Jones (2002, p. 93) has argued convincingly that this is not a convincing assumption. In the post WWII period, the number of scientists and engineers has increased dramatically in the advanced economies. Nevertheless, the growth rate of total factor productivity has not increased in parallel.

Additional Reference:

Jones, C. (2002) *Introduction to Economic Growth*, 2nd edition, New York (W.W. Norton Company).

Kremer, M. (1993) "Population Growth and technological Change: One Million B.C. to 1990", *Quarterly Journal of Economics* 108, 681-716.

Romer, P. (1990) "Endogenous Technological Change", *Journal of Political Economy* 98, S71 - S102.

**Exercise 4.7 (pp. 133-139; Transition Dynamics and Convergence in the Solow Model):**

A useful starting point is the Solow model with exogenous technical progress, containing the following ingredients:

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$$

$$\frac{\dot{N}_t}{N_t} = n$$

$$\frac{\dot{A}_t}{A_t} = g$$

$$\dot{K}_t = I_t - \delta K_t = sY_t - \delta K_t$$

(a) Derive formally the transitional dynamics and the convergence feature of Solow´s baseline model of economic growth.
(b) Analyse the convergence dynamics in the case $\alpha = 1$, known as *AK* model.
(c) Determine the speed of convergence for the following benchmark parameter values: $(1-\alpha) = 0.333$, $g = 0.02$, $n = 0.01$, $\delta = 0.05$. Interpret the convergence speed obtained when plugging in the parameters.

<u>Solution:</u>

<u>Subquestion (a):</u> Define the capital-output ratio as $x_t = K_t / Y_t$. The dynamics of $x_t$ can be derived as follows:

(4.34) $$\frac{\dot{Y}_t}{Y_t} = g + \alpha \frac{\dot{K}_t}{K_t} + (1-\alpha)n$$

(4.35) $$\frac{\dot{K}_t}{K_t} = \frac{s}{x_t} - \delta$$

Given $\frac{\dot{x}_t}{x_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{Y}_t}{Y_t}$, this yields

(4.36) $$\begin{aligned} &\frac{\dot{x}_t}{x_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{Y}_t}{Y_t} \\ \Leftrightarrow &\frac{\dot{x}_t}{x_t} = \left(\frac{s}{x_t} - \delta\right) - \left(g + \alpha\left(\frac{s}{x_t} - \delta\right) + (1-\alpha)n\right) \\ \Leftrightarrow &\frac{\dot{x}_t}{x_t} = (1-\alpha)\left(\frac{s}{x_t} - \frac{g}{1-\alpha} - n - \delta\right) \end{aligned}$$

The steady state value of $x_t$ ,which we label $\tilde{x}$, satisfies $\frac{\dot{x}_t}{x_t} = 0$. This implies

(4.37) $$\tilde{x} = \frac{s}{\frac{g}{1-\alpha} + n + \delta}$$

Equation (4.37) may then equivalently be written as

$$\frac{\dot{x}_t}{x_t} = (1-\alpha)\left(\frac{g}{1-\alpha} + n + \delta\right)\left(\frac{\frac{s}{x_t} - \frac{g}{1-\alpha} - n - \delta}{\frac{g}{1-\alpha} + n + \delta}\right)$$

(4.38) $$\Leftrightarrow \frac{\dot{x}_t}{x_t} = (1-\alpha)\left(\frac{g}{1-\alpha} + n + \delta\right)\left(\frac{1}{x_t}\frac{s}{\frac{g}{1-\alpha} + n + \delta} - 1\right)$$

$$\Leftrightarrow \frac{\dot{x}_t}{x_t} = (1-\alpha)\left(\frac{g}{1-\alpha} + n + \delta\right)\left(\frac{\tilde{x}}{x_t} - 1\right)$$

$$\Leftrightarrow \frac{\dot{x}_t}{x_t} = (1-\alpha)\left(\frac{g}{1-\alpha} + n + \delta\right)\left(\frac{\tilde{x} - x_t}{x_t}\right)$$

Thus, the convergence speed in the model is $\beta = (1-\alpha)\left(\frac{g}{1-\alpha} + n + \delta\right)$. Having derived the dynamics of the capital-output ratio, it is now pretty easy to also derive the dynamics of output per worker. The production function yields

(4.39) $$\frac{\dot{Y}_t}{Y_t} - \frac{\dot{L}_t}{L_t} = \frac{1}{1-\alpha}\frac{\dot{A}_t}{A_t} + \frac{\alpha}{1-\alpha}\frac{\dot{x}_t}{x_t}$$

Substituting in the growth rate of *x* from equation (4.38), we get

(4.40) $$\frac{\dot{Y}_t}{Y_t} - \frac{\dot{L}_t}{L_t} = \frac{g}{1-\alpha} + \alpha\left(\frac{g}{1-\alpha} + n + \delta\right)\left(\frac{\tilde{x} - x_t}{x_t}\right)$$

The immediate implication of equation (4.40) is that the growth rate of output per worker equals the steady state growth rate $\frac{g}{1-\alpha}$ plus or minus the second term due to the capital-output ratio converging towards its steady state level. In other words, rich and poor countries which are similar, in the sense that they possess the same structural parameters and the same production function, should converge (absolute convergence hypothesis).

<u>Subquestion (b):</u> $\beta = (1-\alpha)\left(\frac{g}{1-\alpha} + n + \delta\right)$ implies that there are no transitional dynamics for $\alpha$ = 1, and the growth rate of output per worker is constant.

<u>Subquestion (c):</u> $\beta = (1-0.3333)\left(\frac{0.02}{1-0.3333} + 0.01 + 0.05\right) \approx 0.06$. The interpretation of $\beta$ is as follows: $\tau \times 100\%$ of the divergence between $x_t$ and $\tilde{x}$ will be eliminated after a time interval $t_\tau \equiv -(1/\beta)\log(1-\tau)$ . The solution to the differential equation (4.38) is denoted by

$x_\tau - \tilde{x} = (x_0 - \tilde{x})e^{-\beta\tau}$ . Hence for the half-life of the divergence, we have $1/2 = (x_{\tau_{0.5}} - \tilde{x})/(x_0 - \tilde{x}) = e^{-\beta\tau_{0.5}}$ and the half-life, denoted by $\tau_{0.5}$, has the value: $\tau_{0.5}$ = log 2/$\beta$ = 0.693/$\beta$. For $\beta$ = 0.06 we therefore obtain $\tau_{0.5}$ = 11.55. In other words, convergence is very fast, at least from the perspective of the empirical growth literature. As Barro and Sala-i-Martin (2002, pp. 58-59) have indicated, this estimate is far too high to accord with the empirical evidence suggesting that $\beta$ is more likely in the range of 0.02.

Additional Reference:

Barro, R.J. and X. Sala-i-Martin (2002) *Economic Growth*, 2nd edition, Cambridge (MIT Press).

**Exercise 4.8 [pp. 159-161; Barro (1990)]:**

Consider the production function

$$Y_i = G^{1-\alpha} K_i^\alpha A N_i^{1-\alpha} ,$$

where $G$ is the flow of productive government services. Suppose that the government runs a balanced budget

$$G = \tau Y ,$$

where $\tau$ is the income tax rate. In addition, the utility-maximising behaviour of infinitely-lived households leads to the (standard) Euler equation (see pp. 141-143)

$$\gamma \equiv \frac{\dot{c}}{c} = \frac{1}{\sigma}(f'(k) - \rho) = (r - \rho),$$

where $\rho$ denotes the discount rate.

(a) Characterise and interpret the production function.
(b) Derive the relation between the size of the government and the per capita growth rate. What efficiency condition for the size of productive government services is implied by the set-up?

Solution:

Subquestion (a): The production function implies that each firm $i$ exhibits constant returns to scale in the private inputs $K_i$ and $N_i$. Furthermore, for fixed $G$ the economy is characterised by diminishing returns. If, however, productive infrastructure $G$ rises along with $K$, then the economy is capable of endogenous growth as in the $AK$ model. The underlying reason for this feature is that government services are assumed to be public goods, which can be spread costlessly across additional users.

Subquestion (b): The after-tax profit of firm $i$ is

(4.41) $$(1-\tau)G^{1-\alpha} K_i^\alpha A N_i^{1-\alpha} - wN_i - (r+\delta)K_i ,$$

where $(r + \delta)$ is the rental rate, and $w$ is the wage. Suppose that the rental rate equals the after-tax marginal product of capital, i.e.

(4.42) $$(r+\delta) = (1-\tau)\frac{\partial Y_i}{\partial K_i} = (1-\tau)\alpha A\left(\frac{K_i}{N_i}\right)^{-(1-\alpha)} G^{1-\alpha} .$$

The production function and the balanced budget constraint imply

(4.43) $$G = (\tau AN)^{1/\alpha}\left(\frac{K}{N}\right)$$

Substituting (4.43) in (4.42) and assuming identical firms yields

(4.44) $$(r+\delta) = (1-\tau)\alpha\, A^{1/\alpha}(\tau N)^{(1-\alpha)/\alpha}\ .$$

As in the *AK* model there are no transitional dynamics and the growth rates of all variables are equal to the same constant, $\gamma$. We can determine this constant from the Euler equation for consumption:

(4.45) $$\gamma = \frac{1}{\sigma}\left[(1-\tau)\alpha\, A^{1/\alpha}(\tau N)^{(1-\alpha)/\alpha} - \delta - \rho\right]$$

The effects of productive government services upon growth involve two channels: The term (1-$\tau$) represents the (linear) negative impact of taxation upon the marginal product of capital. The term $\tau^{(1-\alpha)/\alpha}$ represents the (nonlinear) positive effect of infrastructure services. If the government increases the tax rate and keeps piling on extra productive government goods, then the increases in output will taper off due to the diminishing returns. Diminishing returns imply that beyond some threshold the growth-reducing impact of additional taxes will outstrip the productivity-enhancing effects of productive government services. The maximum growth rate can be calculated by setting the derivative of (4.45) with respect to $\tau$ equal to zero:

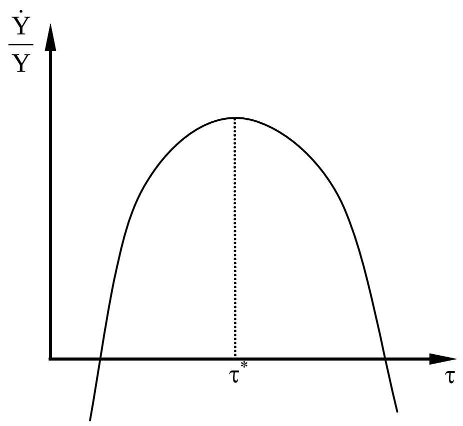
(4.46) $$\frac{\partial\gamma}{\partial\tau} = -\frac{A^{1/\alpha}(\alpha+\tau-1)(\tau N)^{(1-\alpha)/\alpha}}{\sigma\tau} = 0\ .$$

The result is

(4.47) $$\tau^* = \left(\frac{G}{Y}\right)^* = 1-\alpha\ .$$

Equation (4.47) determines the natural efficiency condition for the size of the government. The figure below provides a graphical illustration. For $\tau < \tau^*$ the positive effect of government spending dominates, while for $\tau < \tau^*$ the adverse impact of distortionary taxation gains the upper hand.

**Figure: Government and Growth in the Barro (1990) Model**



Additional Reference:

Barro, R. (1990) "Government Spending in a Simple Model of Endogenous Growth", *Journal of Political Economy* 98, S103-S125.

**Exercise 4.9 [p. 156; Exercise 40]:**

Consider the production function

$$Y_t = A_t K_t^{\alpha} H_t^{1-\alpha} ,$$

where $K_t$ is physical capital and $H_t$ is human capital, respectively. Both types of capital are accumulated according to

$$\dot{K}_t = s_K Y_t - \delta K_t$$

and

$$\dot{H}_t = s_H Y_t - \delta H_t$$

where $s_K$ ($s_H$) are the (exogenous) saving rates of physical (human) capital, and the same depreciation rate ($\delta$) has been assumed for both types of capital without loss of generality.

(a) Calculate the steady state growth rate of output allowing for both types of accumulable factors and interpret the result.
(b) What does the steady state growth path look like for $\alpha = 1$?
(c) Assume that $K$ and $H$ are perfectly substitutable and the competitive firm has the following profit function

$$\pi = A_t K_t^{\alpha} H_t^{1-\alpha} - r_K K - r_H H ,$$

where $r_K$ and $r_H$ are rental rates for $K$ and $H$ respectively. Obtain the steady state growth rates.

Solution:

Subquestion (a): Defining the physical (human) capital-output ratios as $x_K = \frac{K}{Y}$ and $x_H = \frac{H}{Y}$, we obtain

(4.48) $$\frac{\dot{K}}{K} = \frac{s_K}{x_K} - \delta$$

and

(4.49) $$\frac{\dot{H}}{H} = \frac{s_H}{x_H} - \delta .$$

A balanced growth path requires $\frac{\dot{K}}{K} = \frac{\dot{H}}{H}$. We therefore have $\frac{s_K}{x_K} = \frac{s_H}{x_H} \Leftrightarrow \frac{K}{H} = \frac{s_K}{s_H}$. In order to derive the steady state growth path, we must first rewrite the production function

$$
\begin{aligned}
& Y = AK^{\alpha} H^{1-\alpha} \\
& \Leftrightarrow Y = A(x_K Y)^{\alpha} (x_H Y)^{1-\alpha} \\
& \Leftrightarrow 1 = A x_K^{\alpha} x_H^{1-\alpha} \\
& \Leftrightarrow \frac{\dot{A}}{A} = -\alpha \frac{\dot{x}_K}{x_K} - (1-\alpha) \frac{\dot{x}_H}{x_H}
\end{aligned} \tag{4.50}
$$

Hence, positive steady state growth implies that the weighted average of the two capital-output ratios is negative. This, however, would imply exploding values for the growth rates of *H* and *K*. Clearly this is inconsistent with a balanced growth path. Looking at our problem from another perspective, we know that a balanced growth path requires $\frac{\dot{A}}{A} = 0$. Hence, we have

$$
\begin{aligned}
& Y = AK^{\alpha} \left( \frac{s_H}{s_K} K \right)^{1-\alpha} \\
& \Leftrightarrow Y = A \left( \frac{s_H}{s_K} \right)^{1-\alpha} K \\
& \Leftrightarrow \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \left( \frac{s_K Y}{K} \right) - \delta = A s_K^{\alpha} s_H^{1-\alpha} - \delta \ .
\end{aligned} \tag{4.51}
$$

In other words, the steady state growth rate depends positively on both saving rates, and negatively upon the depreciation rate. Furthermore, the *level* of *A* has an effect upon the growth rate. Compared with the baseline Solow model, the *AK*-type model thus presents a fundamentally different picture of economic growth. The distinction is that policy measures leading to changes in the saving rates and/or *A* have *permanent* effects upon the steady state growth rate.

Subquestion (b): For $\alpha = 1$ we obtain $\frac{\dot{Y}}{Y} = A s_K - \delta$ .

Subquestion (c): The first-order-conditions for *K* and *H* are denoted by the following:

$$
\pi_K = \alpha A_t K_t^{\alpha-1} H_t^{1-\alpha} - r_K = 0 \tag{4.52}
$$

$$
\pi_H = (1-\alpha) A_t K_t^{\alpha} H_t^{-\alpha} - r_H = 0 \ . \tag{4.53}
$$

The perfect substitution assumption implies that the rental rates for *K* and *H* are identical:

$$
\begin{aligned}
& \alpha A_t K_t^{\alpha-1} H_t^{1-\alpha} = (1-\alpha) A_t K_t^{\alpha} H_t^{-\alpha} \\
& \Leftrightarrow \alpha H_t = (1-\alpha) K_t \\
& \Leftrightarrow \frac{H}{K} = \frac{1-\alpha}{\alpha} .
\end{aligned} \tag{4.54}
$$

Substituting (4.54) into $\dot{K}_t = s_K Y_t - \delta K_t$ gives the growth rate for *K*,

$$
\begin{aligned}
&\frac{\dot{K}}{K} = s_K \frac{Y}{K} - \delta \\
(4.55) \qquad &\Leftrightarrow \frac{\dot{K}}{K} = s_K A\left(\frac{H}{K}\right)^{1-\alpha} - \delta \\
&\Leftrightarrow \frac{\dot{K}}{K} = s_K A\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} - \delta,
\end{aligned}
$$

which again is a *AK*-type model framework, since the levels of the right hand side variables have an impact upon the balanced growth rate.

**Exercise 4.10 [Derivation of (4.17) via intertemporal constraints; Chapter 2 of Romer (2004)]**

Individuals maximise the intertemporal utility function

$$U = \int_0^{\infty} u(c_t) e^{-\rho t} dt ,$$

subject to the following aggregate capital accumulation equation

$$\frac{dK}{dt} = F(K) - C ,$$

where $K$ is the aggregate capital stock, $F$ is the production function, $\rho$ is the time preference rate, $C$ is the aggregate consumption, and $c = C/N$ is per capita consumption. $N$ is the population of the economy.

(a) Show that the capital accumulation equation can be transformed into the following intertemporal constraint for the economy:

$$\int_0^{\infty} c_t N_t e^{-rt} dt = K(0) + \int_0^{\infty} w_t N_t e^{-rt} dt ,$$

where $r$ is the real interest rate, and $w = (F(K) - rK)/N$ is the per capita income. Assume that the usual transversality condition $\lim_{t \to \infty} A(t) e^{-rt} = 0$ holds.

(b) Show that for $N = \overline{N}$ maximising $U$ subject to the intertemporal constraint of the economy yields equation (4.17) in the textbook.

(c) With the existence of government expenditure and lump-sum taxation, derive the intertemporal budget constraints for the economy, the government, and individuals. Demonstrate that in this setup Ricardian equivalence holds, i.e. the government's choices on to finance its purchases have no impact upon the economy. In contrast, the path of government expenditure affects the path of aggregate consumption.

Solution:

Subquestion (a): The capital accumulation equation can be written as following:

(4.56)
$$\frac{dK}{dt} = F(K) - C$$
$$\Leftrightarrow \frac{dK}{dt} = F(K) - rK + rK - C$$
$$\Leftrightarrow \frac{dK}{dt} = wN + rK - cN$$

Multiplying both sides of the above equation by $e^{-rt}dt$ gives

(4.57)
$$e^{-rt}dK - rKe^{-rt}dt = wNe^{-rt} - cNe^{-rt}$$
$$\Leftrightarrow d\left(e^{-rt}dK\right) = wNe^{-rt} - cNe^{-rt}$$

Integrating both sides of (4.57) from $t = 0$ to $t = \infty$ gives

(4.58)
$$\int_0^\infty d\left(Ke^{-rt}\right) = \int_0^\infty (wN - cN)e^{-rt}dt$$
$$\Leftrightarrow Ke^{-rt}\Big|_0^\infty = K(\infty)e^{-r\infty} - K(0) = \int_0^\infty wNe^{-rt}dt - \int_0^\infty cNe^{-rt}dt$$

From the transversality condition $\lim_{t\to\infty} A(t)e^{-rt} = 0$, we have the intertemporal constraint for the economy:

(4.59)
$$\int_0^\infty c_t N_t e^{-rt}dt = K(0) + \int_0^\infty w_t N_t e^{-rt}dt.$$

Subquestion (b): The intertemporal maximisation problem can be represented by the following Lagrangian:

(4.60)
$$L = \int_0^\infty u(c_t)e^{-\rho t}dt + \lambda\left(K(0) + \int_0^\infty w_t Ne^{-rt}dt - \int_0^\infty c_t Ne^{-rt}dt\right).$$

where $\lambda$ is the Lagrangian multiplier and $N$ is constant by assumption. The first-order condition for $c$ gives:

(4.61)
$$u'(c_t)e^{-\rho t} = \lambda Ne^{-rt}.$$

Take the log of both sides of the above equation and differentiate with respect to time. We then have the following relationship

(4.62)
$$\frac{1}{u'}\frac{du'}{dt} = \rho - r.$$

Using the chain rule, we have

(4.63)
$$\frac{du'}{dt} = \frac{d\,\frac{du(c_t)}{dc_t}}{dt} = \frac{d^2u(c_t)}{dc_t^2}\frac{dc_t}{dt} = u''\frac{dc_t}{dt}.$$

Substituting the above equation into (4.62) gives (4.17) in the textbook,

(4.64) $$\frac{dc_t}{dt} = \left(\frac{u'}{-u''}\right)\left(f'(k) - \rho\right),$$

where $f'(k) = r$ .

<u>Subquestion (c):</u>

With the existence of the government expenditure, $G$, the aggregate capital accumulation path of the economy becomes

(4.65) $$\frac{dK}{dt} = F(K) - C - G \ .$$

Following the same procedure as in the previous section, (b), we have the new intertemporal constraint for the economy,

(4.66) $$\int_0^\infty c_t N_t e^{-rt} dt = K(0) + \int_0^\infty (w_t - g_t) N_t e^{-rt} dt$$

where $g = G/N$ is the government expenditure per person. The financing of the government expenditure is governed by the following process,

(4.67) $$\frac{dB}{dt} = rB + G - T \ ,$$

where $B$ denotes the real value of bonds or debt and $T$ is the total tax revenue. It is assumed that $\lim_{t\to\infty} B(t)e^{-rt} = 0$ . Therefore, the corresponding intertemporal constraint for the government is represented by

(4.68) $$\int_0^\infty t_t N_t e^{-rt} dt = B(0) + \int_0^\infty g_t N_t e^{-rt} dt \ ,$$

where $t = T/N$ is per capita taxation and $B(0)$ is the government total debt at $t$ = 0. Equations (4.66) and (4.68) yield consumers' intertemporal budget constraint:

(4.69) $$\int_0^\infty c_t N_t e^{-rt} dt = K(0) + B(0) + \int_0^\infty (w_t - t_t) N_t e^{-rt} dt \ .$$

Assume that $N$ is a constant and we have the individual consumers' budget constraint

(4.70) $$\int_0^\infty c_t e^{-rt} dt = \frac{K(0)}{N} + \frac{B(0)}{N} + \int_0^\infty (w_t - t_t) e^{-rt} dt \ .$$

The value of bonds that a consumer holds and the taxation affect his or her intertemporal budget constraint, according to (4.70). However, for the economy as a whole, the way in which the government finances its total expenditure does not affect the economy's constraints (4.65) and (4.66) for determining consumption. As long as the government must repay its debt eventually – the government budget constraint (4.68) holds – how the government finances its expenditure has no impact on the consumption, which is the proposition of Ricardian equivalence. In contrast, the level of expenditures does have a real impact on consumption.

## II. Software Tools

**Software-Exercise 4.1: Determine the saddle paths and steady states for consumption and capital in equations (4.17) and (4.18) on p. 141 of the textbook for a *CRRA* utility function and a *CD* production function using the Backward Integration Technique**

Allowing for depreciation of the capital stock, (4.17) and (4.18) are modified as follows:

$$(4.71) \quad \frac{dc}{dt} = -\frac{u'(c)}{u''(c)}\left(f'(k) - \delta - \rho\right),$$

and

$$(4.72) \quad \frac{dk}{dt} = f(k) - c - \delta k,$$

where $c$ is consumption, $k$ denotes the capital stock, $u$ is the immediate utility, $f$ is production function, $\delta$ is the depreciation rate of capital, and $\rho$ is the time preference rate. Assume a *CRRA* utility function as in exercise 4.3 and use the production $f = k^a$. This yields

$$(4.73) \quad \frac{dc}{dt} = \frac{c}{\sigma}\left(ak^{a-1} - \delta - \rho\right)$$

and

$$(4.74) \quad \frac{dk}{dt} = k^a - c - \delta k,$$

where $\sigma$ denotes the risk elasticity of *CRRA* utility. Demonstrate the impacts of changes in $\rho$, $\sigma$, and $\delta$ on saddle paths of consumption and capital over time.

<u>Solutions</u>:

Equations (4.73) and (4.74) are a saddle-path stable dynamic differential system with initial values of $k$ and $c$. Note that once the initial value of $k$ is given, there only exists one corresponding value for the initial value of $c$ leading to the new equilibrium along the stable saddle path. Any other initial values of $c$ (given $k$) will leads to a divergence of $k(t)$ and $c(t)$ over time.

Algorithms have been developed which can solve such saddle path problems. Here we adopt the backward integration algorithm suggested by Brunner and Strulik (2001). Three "m" files are used to compute the processes of consumption and capital over time and their corresponding phase diagram: "fun_Exercise_4_1_ramsdot2.m" lists the equations (4. ) and (4. ) for the input function for the main m file: "main_Exercise_4_1_ramsey_1f.m", which uses "ode45mod.m" from Brunner and Strulik (2001) to solve the system.

The input data are as follows:

```
% beginning of input data for benchmark
delta   = .05;  % depreciation rate of k
rho     = .07;  % rate of time preference
a       = .3;   % capital share
```

```
sigma    = 1;     % risk elasticity for CRRA
kfinal   = 1.0;   % initial value of k
choiceofplot = 3;   % "1" for plot of the impact of rho;
                    % "2" for plot of the impact of sigma;
                    % "3" for plot of the impact of delta;
% end of input data for benchmark
```

Given the choice of the value of choiceofplot, 1 or 2 or 3, the matlab m file will generate diagrams that shows the effect of $\rho$ or $\sigma$ or $\delta$ on the paths of consumption and capitals and corresponding phase diagram up to steady-state point, indexed by "*".

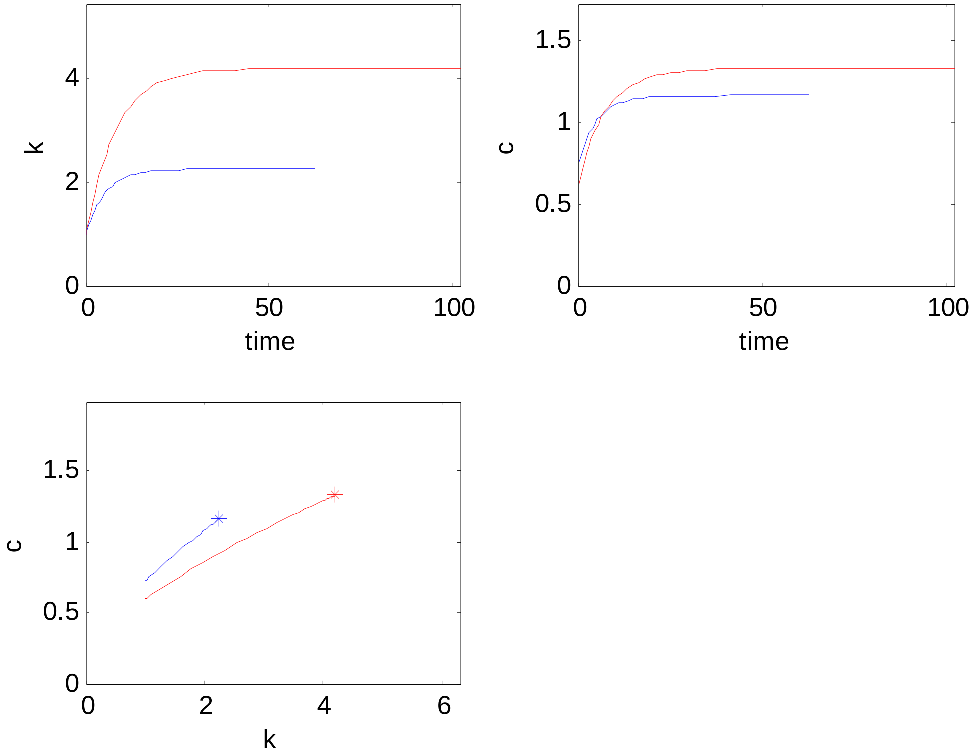
*The effect of* $\rho$

The range of $\rho$ is controlled in line 31 of main m file:

```
rho = 0.12 - 0.06*(i-1); % range of rho
```

which means that the m file runs for the value of $\rho$ = 0.12 and $\rho$ = 0.06 (red line). The following diagrams demonstrating how the economy converges to the steady state are generated.

**Software exercise 4.1: the effect of rho**



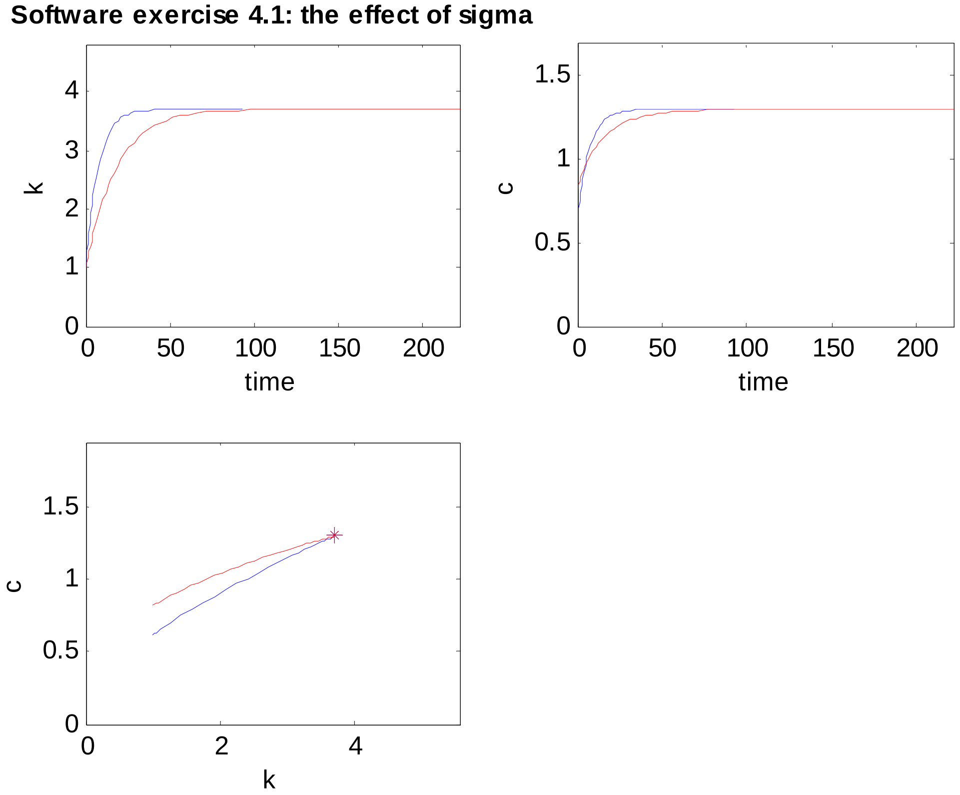

A higher $\rho$ leads to lower equilibrium values of $k$ due to equation (4. ). We can see that a fall in $\rho$ leads to a jump in $c$ – consumers care about the future and save more immediately. After that, consumption approaches new steady-state consumption.

*The effect of* $\sigma$

The range of $\sigma$ is controlled in the line 33 of main m file:

```
sigma = 1.0 + 3.0*(i-1); % range of sigma
```

which means that the m file runs for the value of $\sigma = 1.0$ and $\sigma = 4.0$ (red line). The following diagrams are generated.




A higher $\sigma$ mainly affects the adjustment speed of *k* and *c*, but not the steady state values. A lower $\sigma$ leads directly to a higher adjustment speed for consumption, and thus indirectly to a higher adjustment speed for capital.

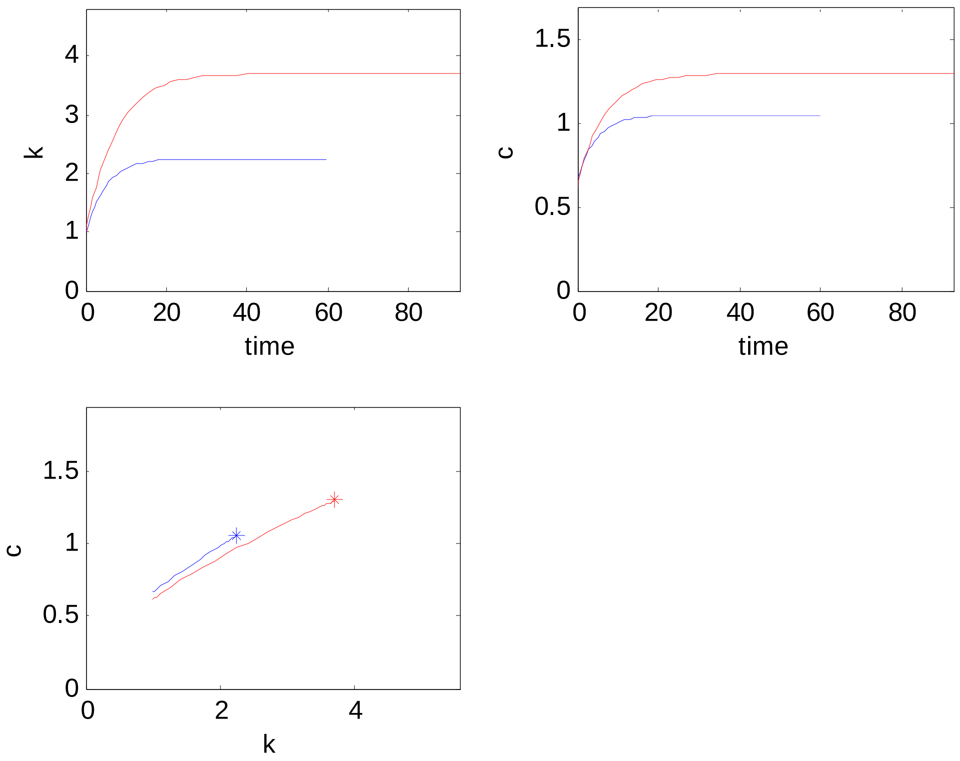
*The effect of* $\delta$

The range of $\delta$ is controlled in the line 35 of main m file:

```
delta = 0.10 - 0.05*(i-1); % range of delta
```

which means that the m file runs for the value of $\delta = 0.1$ and $\rho = 0.05$ (red line). The following diagrams are generated.

**Software exercise 4.1: the effect of delta**




Governed by equations (4.73) and (4.74), higher $\delta$ values lead to lower equilibrium values of *k* and *c* respectively.

<u>Additional References:</u>

Brunner, M. and Strulik, H. (2002) "Solution of Perfect Foresight Saddlepoint Problems: A Simple Method and Applications", *Journal of Economic Dynamics and Control*, 26, pp. 737-753.

Brunner, M. and Strulik, H. (2001) "Code for "Solution of Perfect Foresight Sattlepoint Problems: A Simple Method and Applications"; see [http://kaldor.vwl.uni-hannover.de/holger/software/index.php](http://kaldor.vwl.uni-hannover.de/holger/software/index.php).

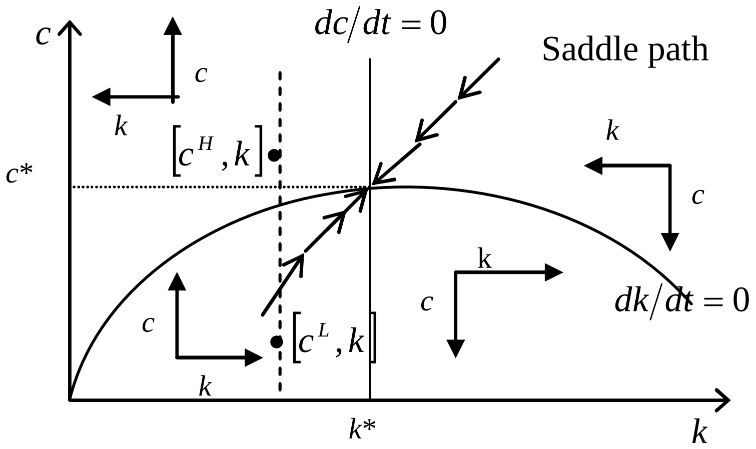
**Software-Exercise 4.2: Solve Equations (4.73) and (4.74) by a Forward Shooting Algorithm.**

<u>Solutions:</u>

For the saddle path system of (4.73) and (4.74), with the given initial value of *k*, there only exists one value of *c* so that [*c*, *k*] can move to the steady state point $[c^*, k^*]$ in the phase diagram. Let's index $[c^s, k^s]$ as a point on the saddle path of the system and $c^s < c^*$, $k^s < k^*$. For the given $k^s$, any values of consumption other than $c^s$ lead to points violating the budget constraints or transversality conditions. Thus, we can use the following simple forward multi-shooting algorithm to obtain $[c^s, k^s]$:

(a) Guess a high value $c^H$ and a low value $c^L$ for consumption and let $c^1 = (c^L + c^H)/2$.
(b) Solve the differential system of (4.73) and (4.74) with initial condition $[c^1, k^s]$, i.e. roll the system forward in time. Stop the system when $dc/dt < 0$ or $dk/dt < 0$ and index it as $[c(T), k(T)]$.

(d) If $\left|c(T)-c^*\right|<\varepsilon$, where $\varepsilon$ denotes a small value, then $c^s=c^1$ for given $k^s$ and we have the answer; otherwise, set $c^L=c^1$ for $dc/dt<0$ or $c^H=c^1$ for $dk/dt<0$ and go back to (a). Note that $dc(T)/dt<0$ occurs for the case where the initial value of $c$ is too low and $dk(T)/dt<0$ occurs for the case where the initial value of $c$ is too high.



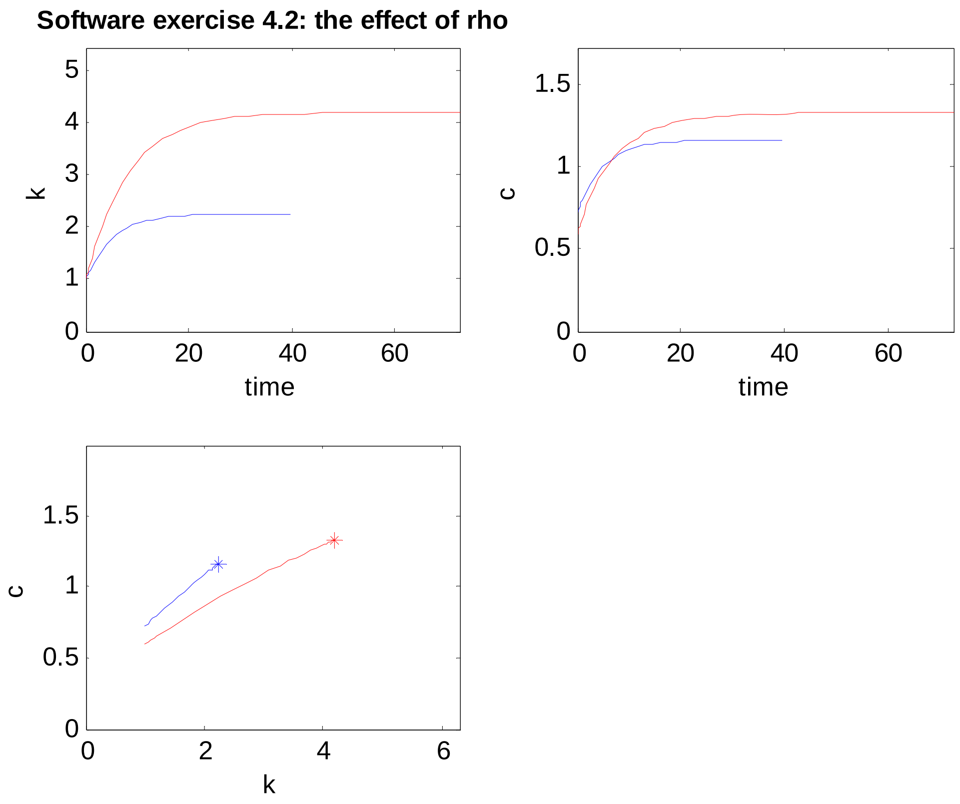

Three “m” files are used to compute the processes of consumption and capital over time and their corresponding phase diagram by the above forward shooting algorithm: “fun_Exercise_4_2_ramsdot2.m”, “fun_Exercise_4_2_events.m”, and “main_Exercise_4_2.m”.

The input data are as the same as in previous exercise:

```
% beginning of input data for benchmark
rho      = 0.07;     % rate of time preference
a        = 0.3;      % capital share
sigma    = 1;        % risk elasticity for CRRA
delta    = 0.05;     % depreciation rate of k
k0       = 1.0;      % initial value of k
choiceofplot = 1;    % "1" for plot of the impact of rho;
                     % "2" for plot of the impact of sigma;
                     % "3" for plot of the impact of delta;
% end of input data for benchmark
```

The program is based upon Judd (1998), chapter 10, p. 357 and employs some code from Francesco Franco´s "shoot.m" MATLAB code (see http://web.hec.gov.pk/OcwWeb/Economics/14-451Macroeconomic-Theory-ISpring2003/Assignments/index.htm).

Given the choice of the value of choiceofplot, 1 or 2 or 3, the matlab m file will generate diagrams that show the effect of $\rho$ or $\sigma$ or $\delta$ on the paths of consumption and capitals and corresponding phase diagram up to steady-state point, indexed by “*”. The results are identical to those of the previous exercise. For example, the effect of $\rho$ is shown by the following figures:




Additional References:

Goffe, W.L. (1993) “A User´s Guide to the Numerical Solution of Two-Point Boundary Value Problems Arising in Continuous Time Dynamic Economic Models”, *Computational Economics* 6, pp. 249-255.

Judd, K.L. (1998) *Numerical Methods in Economics*, Cambridge (MIT Press).

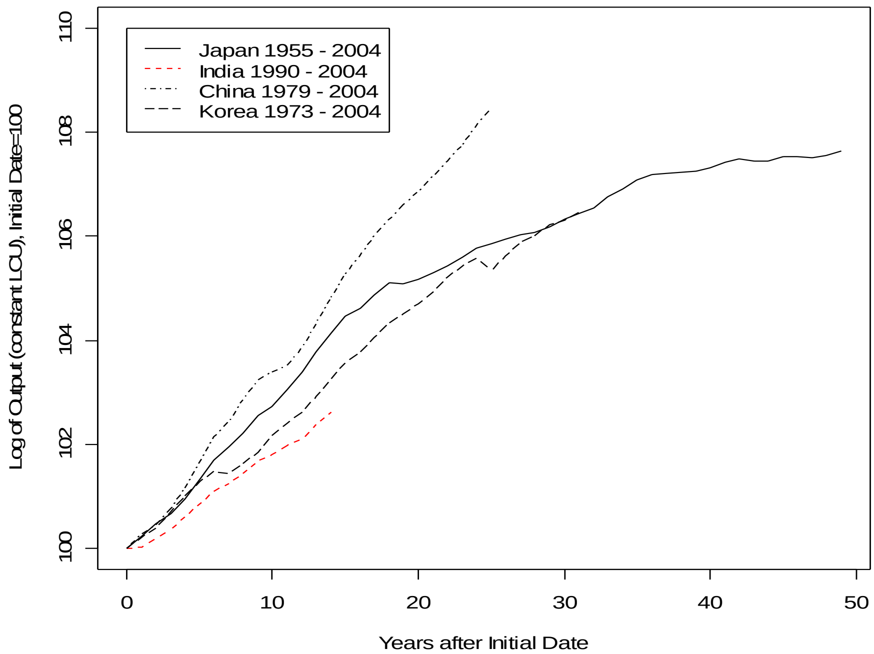
Stemp, P.J. and R.D. Herbert (2006) “Solving Non-Linear Models with Saddle-Path Instabilities”, *Computational Economics* 28, pp. 211-231.

**Software Exercise 4.3: China´s Growth Compared with Previous Historical Episodes in Asia**

After a long period of isolation, China began to embrace reform and globalisation at the end of the 1970s. Since then China´s GDP has grown at an annual rate of over 9 percent for many years. To put China´s GDP growth into perspective, compare China´s growth experience with previous episodes of rapid growth in the “miracle” economies of India, Japan, and Korea (see the attached dataset Asia_Growth.xls, annual GDP data in local currency units, with constant prices). Are there differences and/or similarities across countries at corresponding phases of their integration into the world economy? Does the Chinese catching-up and growth experience stand out as unusual?

Solution:

**GDP Growth Across Countries**




Notes: *t* indicates the year when the integration and sustained growth process began: 1955 for Japan, 1973 for Korea, 1979 for China and 1990 for India. Index beginning of period = 100.

What is most notable is China´s impressive performance. The graph indicates that China´s GDP has grown markedly and its growth momentum is above that of the other three countries at corresponding phases of their development. On the other hand, the growth experience of the remaining three countries in their episodes of fast integration and catching-up was largely commensurate and exhibits many similarities.

**Software Exercise 4.4: TFP Across Countries and the European Productivity Malaise (pp. 149-152).**

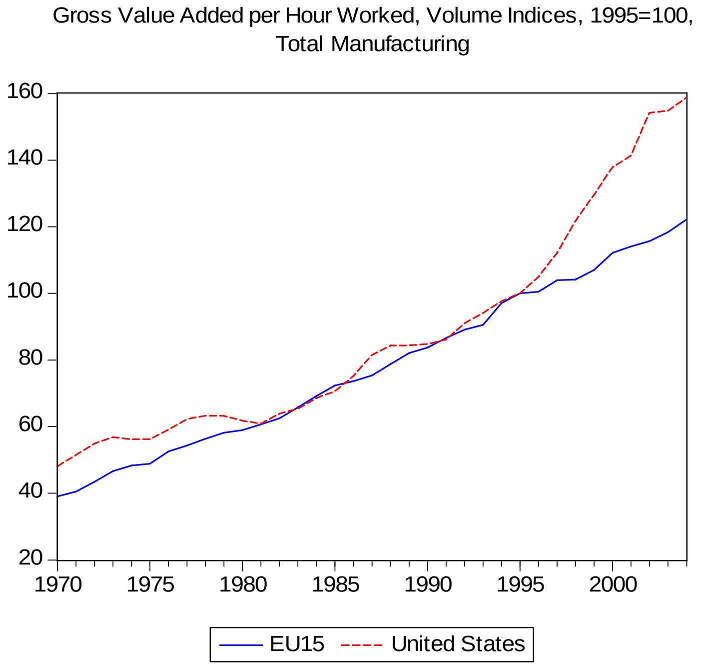
The rise in productivity determines how fast living standards can improve. Unfortunately, while productivity growth is one of the most important economic indicators, it is also one of the most mysterious. After growing strongly in the 1960s, productivity growth decelerated in the 1970s and 1980s. This slowdown has been attributed, among other things, to demographic shifts, oil price increases, slower growth in investment, and increased government regulations. In the 1980s, economists were faced with another puzzle. Increased use of computers was having little, if any, measurable impact on productivity growth. Nobel Laureate Robert Solow summed this puzzle up nicely when he said in 1987: "You can see the computer age everywhere but in the productivity statistics". More recently, there has been an acceleration of productivity growth in the US. What is unusual about this acceleration is that it has occurred in the late stage of an expansion, when productivity growth usually slows down. While very good news, this upsurge has led to arguments among academics and policy makers about how pervasive and sustainable this trend is.

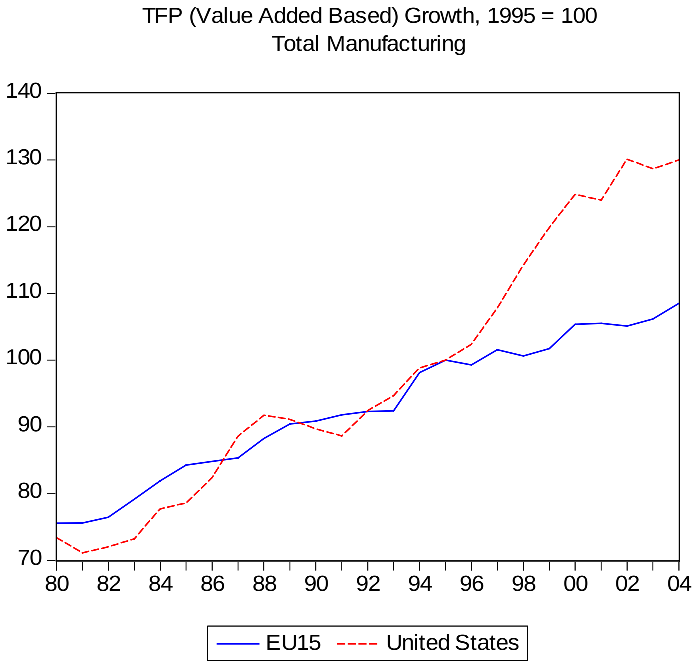
In order to gain an insight into the productivity paradox and debate, draw upon the disaggregated EU-KLEMS database which accounts for the sources of European vs. U.S. growth and productivity between 1970 and 2004 (see http://www.euklems.net/).

(a) Download the time series for labour productivity (gross value added per hour worked, series *LP_I*) and TFP (series *TFPva_I*) for EU15 (old member states) and the US and interpret the historical developments for total manufacturing over time.

(b) Part of the rise in productivity during the last five years may be due to strong growth in investment - mostly in computers and information technology (ICT). Download the time series for ICT capital services (series *CAPIT_QI*) in manufacturing for EU15 and the US and interpret the developments over time. What is the relative importance of this source of TFP growth in recent years across countries?

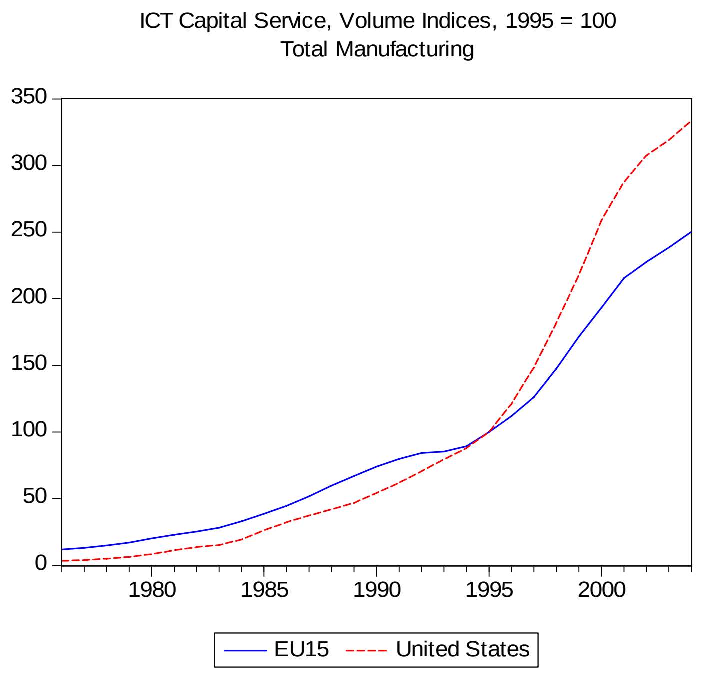
Subquestion (a):




In the period 1995-2004, US productivity and TFP growth picked up markedly - leading to the recent “productivity miracle”. While productivity in the US accelerated, productivity in the EU15 countries stalled. Even more dismal was EU15´s TFP growth. The acceleration of productivity growth in the US has several macroeconomic implications. For example, if the recent rise in productivity growth is sustainable, then the speed limit of the US economy is higher and the Fed can accommodate faster growth without having to worry about inflation.

<u>Subquestion (b):</u>




The data indicate that the US underwent a much bigger shift into new ICT technologies. It also turns out that, with a lag of about the 5 years, computers have been showing up in the productivity statistics. Therefore, the above mentioned Solow paradox and the puzzling surge in US productivity in recent years may be two sides of the same coin. On the contrary, the EU15 countries have fallen behind the US due to lower uptake in ICT intensity.