Study Guide

to accompany

F.-C. Bagliano and G. Bertola
Models for Dynamic Macroeconomics
Oxford University Press, Oxford 2004.

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Chapter 1: Dynamic Consumption Theory

I. Motivational Questions and Exercises:

Exercise 1.1 (p. 3, footnote 1):

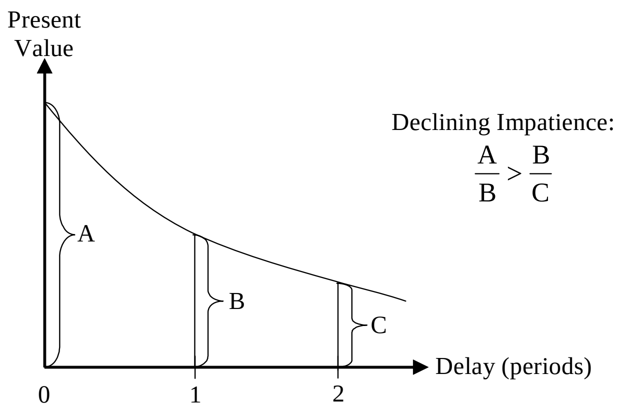
In what way does hyperbolic discounting differ from exponential discounting?

Solution: The conventional exponential discounting model weights utility in year T by the discount factor $F(T)$, where

(1.1) $$F(T) = \delta^T$$

and δ = 1/(1+ρ). A popular alternative to exponential discounting is hyperbolic discounting, which captures the commonly observed anomaly of declining impatience, i.e. the fact that discount factors increase with delay. Figure (1.1) illustrates a hyperbolic discount function with declining impatience. It is apparent that hyperbolic discounting places an especially large weight on immediate payoffs as compared to deferred payoffs.

Figure 1.1: Hyperbolic Discount Function



The hyperbolic formulation for discrete time periods yields discount factors $F(T)$ given by

(1.2) $$\{1\ ,\lambda\delta\ ,\lambda\delta^2, \lambda\delta^3,\ ...\}\ ,$$

where 0 < λ < 1 and δ < 1. The discount factor terms involving δ are all multiplied by the parameter λ except in the first period. Assuming additivity over time, an explicit example of hyperbolic utility is

(1.3) $$u(c_0) + \frac{1}{4}u(c_1) + \frac{1}{8}u(c_2) + \frac{1}{16}u(c_3) + ...$$

An alternative flexible formulation proposed by Loewenstein and Prelec (1992) takes the form

(1.4) $$F(T) = (1 + \rho T)^{-\omega/\rho}$$

where ω, ρ > 0. A well-known problem of hyperbolic discounting is that it induces patterns of inconsistency. What is dynamic consistency? Let´s look at a function to be maximised at some point in the future and once again at a further point in the future. The tail for the initial time that corresponds to the future actions must be the same as that when you consider the weighting of future actions from the later point in time (a difference of a scale factor is allowed). Another way of stating this is that if you maximise over the entire function, you get the same optimal path as you would if you set the first element and then maximize over the rest of your choices. What is more, the ordering of the future paths will be the same independent of whether you have not yet maximised the first element or not. When dynamic consistency does not hold, then one is faced with a rather odd game. It is as if the person playing the game today is different from the person playing the game tomorrow. Assume, for example, the utility function (1.4). From t_1 onwards, the future looks the same as it did from t_0 onwards. Therefore the hyperbolic utility function looks like

$$(1.5) \qquad u(c_1) + \frac{1}{4}u(c_2) + \frac{1}{8}u(c_3) + \ldots$$

which cannot be transformed into the tail from the standpoint of the first period by multiplying the entire function by any scale factor. On the contrary, exponential discounting characterizes the preferences of a rational agent, meaning that the preferences are dynamically consistent.

Additional References:

Laibson, D. (1997) "Golden Eggs and Hyperbolic Discounting", *Quarterly Journal of Economics* 112, 443-477.

Loewenstein, G. and D. Prelec (1992) „Anomalies in Intertemporal Choice: Evidence and an Interpretation“, *Quarterly Journal of Economics* 107, 573-597.

Exercise 1.2:

Consider an American soldier who belongs to a battalion of troops committed to duty in Iraq. He realises that while being stationed in Iraq he may die and therefore discounts his future according to

$$(1.6) \qquad u(c_0) + \frac{1}{4}u(c_1) + \frac{1}{8}u(c_2) + \frac{1}{16}u(c_3) + \ldots$$

Assume that he emerges from his tour of duty in Iraq alive and well. Once he has returned home, from time $t = 1$ onwards, he discounts the future according to

$$(1.7) \qquad u(c_1) + \frac{1}{2}u(c_2) + \frac{1}{4}u(c_3) + \ldots$$

Does dynamic consistency hold for this "going to the front" example?

Solution: Equations (1.6) and (1.7) may look hyperbolic, but they are not. Equation (1.7) can be transformed from (1.6) simply by multiplying by ¼.

Exercise 1.3:

Verify that for the case of the CRRA utility function the absolute value of the elasticity of marginal utility is γ.

Solution: The elasticity of marginal utility is defined as

(1.8) $$\frac{du'(c)}{dc}\frac{c}{u'(c)} = \frac{u''(c)c}{u'(c)} = \frac{\left(-\gamma c^{-\gamma-1}\right)c}{c^{-\gamma}} = -\gamma$$

Exercise 1.4:

Suppose that the parameter of the CRRA utility function is $\gamma = 1$. What happens in this special case?

Solution: Consider the limit

(1.9) $$\lim_{\gamma \to 1} \frac{f(\gamma)}{g(\gamma)}$$

and suppose that $\lim_{\gamma \to 1} f(\gamma)$ and $\lim_{\gamma \to 1} g(\gamma)$ are both zero. L´Hospital´s Rule provides a powerful method for evaluating such limits and to determine indeterminate forms of type 0/0 or ∞/∞. According to L´Hospital´s Rule

(1.10) $$\lim_{\gamma \to 1} \frac{f(\gamma)}{g(\gamma)} = \lim_{\gamma \to 1} \frac{f'(\gamma)}{g'(\gamma)}.$$

Therefore:

(1.11) $$\lim_{\gamma \to 1} \frac{f'(\gamma)}{g'(\gamma)} = \lim_{\gamma \to 1} \left(\frac{-\ln(c) c^{1-\gamma}}{-1} \right) = \ln(c) \quad .$$

Exercise 1.5:

The Arrow-Pratt measure of absolute risk aversion is defined as $-u''(c)/u'(c)$. The higher the curvature of $u(c)$, the higher the risk aversion. Consider an agent with expected utility $E(U) = \kappa u(c_1) + (1-\kappa) u(c_2)$ where c_S is the consumption level in state s = 1,2 and $\kappa\left(1-\kappa\right)$ is the corresponding probability assigned to state s. Provide a justification for the Arrow-Pratt measure of risk aversion by showing that higher values of $-u''(c)/u'(c)$ imply indifference curves which are more highly bowed.

Solution: An indifference curve I defines c_2 as a function of c_1

(1.12) $$c_2 = I(c_1).$$

The slope of the indifference curve in Figure (1.2) is given by

(1.13) $$\frac{dc_2}{dc_1} = \frac{dI(c_1)}{dc_1} = -\frac{\kappa u'(c_1)}{(1-\kappa)u'(c_2)} = -\frac{\kappa u'(c_1)}{(1-\kappa)u'(I(c_1))}.$$

Differentiating (1.13) yields:

(1.14) $$\frac{d^2 I(c_1)}{d\, c_1^2} = -\frac{\kappa}{(1-\kappa)u'(c_2)^2}\left[u''(c_1)u'(I(c_1)) - u'(c_1)u''(I(c_1))\frac{dI}{d\, c_1}\right] \quad .$$

At the 45° degree line $c_1 = c_2 = I(c_1)$. Furthermore, $dI(c_1)/d\,c_1 = -\kappa/(1-\kappa)$, $u''(c_2) = u''(I(c_1)) = u''(c_1)$ and $u'(c_2) = u'(I(c_1)) = u'(c_1)$. Thus, equation (1.14) can be rewritten as

(1.15) $$\frac{d^2 I(c_1)}{d\,c_1^2} = -\frac{u''(c)}{u'(c)}\left[1 + \frac{\kappa}{(1-\kappa)}\right]\frac{\kappa}{(1-\kappa)}.$$

Consider the two indifference curves I_1 and I_2 in Figure (1.2) at point *P*. Since $c_1 = c_2$ the slope of both indifference curves is $d\,I_1/d\,c_1 = d\,I_2/d\,c_1 = -\kappa/(1-\kappa)$. On the other hand, we have

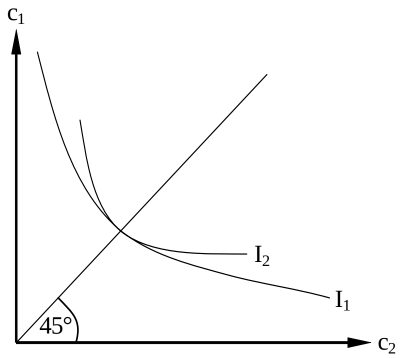
(1.16) $$\frac{d^2 I_2}{d\,c_1^2} > \frac{d^2 I_1}{d\,c_1^2}.$$

The inequality (1.16) is equivalent to

(1.17) $$\frac{-u''_1(c)}{u'_1} < \frac{-u''_2(c)}{u'_2}$$

and therefore an agent´s Arrow-Pratt measure of absolute risk aversion increases when his indifference curves are more bowed inwards, implying that he is less willing to accept risky prospects.

Figure 1.2: Indifference Curves and Arrow-Pratt Measure of Absolute Risk Aversion



Exercise 1.6:

Determine the degree of absolute prudence $P(c) \equiv -u'''(c)/u''(c)$ and relative prudence $p(c) \equiv -u'''(c)c/u''(c)$ for the quadratic utility function $u(c) = c - (b/2)c^2$, $b > 0$, the exponential utility function $u(c) = -1/\gamma\, e^{-\gamma c}$, $\gamma > 0$, and the isoelastic utility function $u(c) = (c^{1-\gamma} - 1)/(1-\gamma)$, respectively.

<u>Solution:</u> Kimball (1990) measures the strength of the precautionary saving motive. His theory of precautionary savings is mathematically analogous to the Arrow-Pratt theory of risk aversion. Kimball measures the absolute prudence by $P(c) \equiv -u'''(c)/u''(c)$, and shows that precautionary saving becomes larger as the value of absolute prudence gets larger.

The derivatives for the three instantaneous utility function are:

(1.18) $$u(c) = c - \frac{b}{2}c^2;\ u'(c) = 1 - bc;\ u''(c) = -b;\ u'''(c) = 0;\ P(c) = 0;\ p(c) = 0.$$

Since $P(c) = p(c) = 0$, agents have no precautionary saving motive. The reason for this is due to the fact that marginal utility is linear and thus unaffected by mean preserving increases in variance.

(1.19) $u(c) = -\frac{1}{\gamma}e^{-\gamma c}$; $u'(c) = e^{-\gamma c}$; $u''(c) = -\gamma\, e^{-\gamma c}$; $u'''(c) = \gamma^2 e^{-\gamma c}$; $P(c) = \gamma$; $p(c) = \gamma c$.

For the exponential utility function, the absolute degree of prudence is constant and unrelated to consumption while the relative degree of prudence is increasing in c.

(1.20) $u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}$; $u'(c) = c^{-\gamma}$; $u''(c) = -\gamma\, c^{-\gamma-1}$; $u'''(c) = \gamma\, c^{-\gamma-2}(\gamma+1)$; $P(c) = \frac{\gamma+1}{c}$; $p(c) = \gamma + 1$.

For the isoelastic utility function, the absolute degree of prudence is decreasing in consumption, while the relative degree of prudence is constant.

Additional Reference:

Kimball, M. (1990) "Precautionary Saving in the Small and the Large", *Econometrica* 58, 53-73.

Exercise 1.7 (p. 5):

Illustrate the derivation of equation (1.5) of the textbook.

Solution: (a) The first (and more general) way is via dynamic programming. It is obvious that we can rewrite the objective function as

(1.21) $$\underset{c_{t+i}}{Max}\, U_t = E_t\left[u(c_t) + \sum_{i=1}^{\infty}\frac{u(c_{t+i})}{(1+\rho)^i}\right].$$

If we re-index $\sum_{i=1}^{\infty}\frac{u(c_{t+i})}{(1+\rho)^i}$ from i = 0, instead of i = 1, we have $\frac{1}{(1+\rho)}\sum_{i=0}^{\infty}\frac{u(c_{t+1+i})}{(1+\rho)^i}$, which can be written as $\frac{1}{(1+\rho)}U_{t+1}(.)$. Thus,

(1.22) $$\underset{c_{t+i}}{Max}\, U_t = E_t\left[u(c_t) + \frac{1}{(1+\rho)}U_{t+1}(.)\right].$$

Since c is the control variable and after the maximization is no longer a function of state variable A, we find that V_t, the intertemporal utility after maximization, is a function of A_t and that V_{t+1} is a function of A_{t+1}. Thus the objective function, in terms of V, becomes:

(1.23) $$V_t(A_t) = \underset{c_t}{Max}\left[u(c_t) + \frac{1}{(1+\rho)}E_t V_{t+1}(A_{t+1})\right],$$

which is the standard Bellman equation for discrete-time cases. The first order condition gives

(1.24) $u'(c_t)+\frac{1}{1+\rho}E_t V'_{t+1}(A_{t+1})\frac{dE_t A_{t+1}}{dc_t}=0$.

We have $\frac{dA_{t+1}}{dc_t}=-1$, according to state equation: $A_{t+i+1}=(1+r)A_{t+i}+y_{t+i}-c_{t+i}$. Thus, we have

(1.25) $u'(c_t)=\frac{1}{1+\rho}E_t V'_{t+1}(A_{t+1})$.

The envelope theorem for $V_t(A_t)\equiv U(c_t(A_t),A_t)$ for optimised c gives

(1.26) $\frac{dV_t(A_t)}{dA_t}\equiv\frac{\partial U(c_t(A_t),A_t)}{\partial A_t}=\frac{\partial u(c_t)}{\partial A_t}=\frac{\partial u((1+r)A_t+y_t-A_{t+1})}{\partial A_t}=u'(c_t)(1+r)$.

The above equation also implies for the period t+1 that

(1.27) $E_t V'_{t+1}(A_{t+1})=E_t u'(c_{t+1})(1+r)$.

Substituting into equation (1.25) gives

(1.28) $u'(c_t)=\frac{1+r}{1+\rho}E_t u'(c_{t+1})$,

which is equation (1.5).

(b) Equation (1.5) can also be derived using the ordinary Lagrangian method. We can construct the Lagrangian for the objective function

(1.29) $L=E_t\sum_{i=0}^{\infty}\frac{u(c_{t+i})}{(1+\rho)^i}+E_t\sum_{i=0}^{\infty}\lambda_i((1+r)A_{t+i}-A_{t+i+1}+y_{t+i}-c_{t+i})$.

It is obvious that we cannot compare the Lagrangian multipliers in different periods due to the depreciation (discounting by r) of A, in terms of values at t. Thus, it is convenient to create an new variable such that

(1.30) $M_i=(1+r)^i\lambda_i$.

The reason for this trick is that the multipliers depreciate at a rate of r. The Lagrangian now becomes

(1.31) $L=E_t\left[\sum_{i=0}^{\infty}\left[\frac{u(c_{t+i})}{(1+\rho)^i}+\frac{M_i}{(1+r)^i}((1+r)A_{t+i}-A_{t+i+1}+y_{t+i}-c_{t+i})\right]\right]$.

The first order condition for the consumer at periods t and t+1 is

(1.32) $u'(c_t)=M_0$,

(1.33) $\frac{E_t u'(c_{t+1})}{1+\rho}=\frac{M_1}{1+r}$.

The multiplier should be comparable, and indeed equal, for the adjusted M so that $M_0 = M_1$. Thus, we obtain equation (1.5):

$$(1.34) \quad u'(c_t) = \frac{1+r}{1+\rho} E_t u'(c_{t+1}).$$

Exercise 1.8 (p. 24):

Derive the Euler equation (1.44).

Solution: Expanding $u'(c_{t+1})$ around the value of c_t for c_{t+1} by Taylor expansion up to the second order yields

$$(1.35) \quad u'(c_{t+1}) = u'(c_t) + u''(c_t)(c_{t+1} - c_t) + \frac{1}{2} u'''(c_t)(c_{t+1} - c_t)^2 .$$

Taking expectations on both sides gives

$$(1.36) \quad E_t u'(c_{t+1}) = u'(c_t) + u''(c_t) E_t (c_{t+1} - c_t) + \frac{1}{2} u'''(c_t) E_t (c_{t+1} - c_t)^2 .$$

By the Euler equation $E_t u'(c_{t+1}) = u'(c_t)$ on p. 24, we then have equation (1.44).

Exercise 1.9 (p. 26):

Derive the first-order condition (1.47).

Solution: This is similar to equation (1.5) and it still uses the same objective function and difference equation. Thus, it must yield the same results as equation (1.5).

$$(1.37) \quad u'(c_t) = \frac{1+r}{1+\rho} E_t u'(c_{t+1}).$$

Substituting equation (1.45) into the above equation and using the assumption $r = \rho$ at the end of p. 25 gives equation (1.47)

$$(1.38) \quad e^{-\gamma c_t} = E_t \left(e^{-\gamma c_{t+1}} \right).$$

Exercise 1.10 (p. 28):

Derive the first-order condition (1.57).

Solution: The lognormal distribution is given by $E\left(e^{bx}\right) = e^{b\mu + \frac{b^2\sigma^2}{2}}$, if $x \sim N\left[\mu, \sigma^2\right]$ (see the Derivation below). Therefore, if $\varepsilon \sim \left[0, \sigma_\varepsilon^2\right]$, then

$$(1.39) \quad E_t\left[e^{-\gamma\theta\varepsilon}\right] = e^{-\gamma\theta 0 + (-\gamma\theta)^2 \sigma_\varepsilon^2 / 2} = e^{\gamma^2\theta^2\sigma_\varepsilon^2/2}$$

Thus,

(1.40) $$K_t = \frac{1}{\gamma}\ln\left(E_t\left[e^{-\gamma\theta\varepsilon}\right]\right) = \frac{1}{\gamma}\ln\left(e^{\gamma^2\theta^2\sigma_\varepsilon^2/2}\right) = \frac{1}{\gamma}\frac{\gamma^2\theta^2\sigma_\varepsilon^2}{2} = \frac{\gamma\theta^2\sigma_\varepsilon^2}{2}$$

Derivation: Suppose a random variable x is normal with mean μ and standard deviation σ. Then,

$$E[\exp(bx)] = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(bx)\exp\left[-\frac{(x-\mu)}{2\sigma^2}\right]dx$$

$$= \frac{\exp(b\mu)}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left[by - \frac{y^2}{2\sigma^2}\right]dy$$

(1.41) $$= \frac{\exp\left(b\mu + \frac{1}{2}b^2\sigma^2\right)}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left[-\frac{\left(y-b\sigma^2\right)^2}{2\sigma^2}\right]dy$$

$$= \frac{\exp\left(b\mu + \frac{1}{2}b^2\sigma^2\right)}{\sigma\sqrt{2\pi}}$$

$$= \exp\left(b\mu + \frac{1}{2}b^2\sigma^2\right).$$

The first line is merely the definition of the expectation using the standard formula for the normal density. The second line transforms this into a new variable $y = x - \mu$ which is also normally distributed. The third line then completes a square, while the fourth uses the fact $y - b\sigma^2$ is also normal so its density integrates out to unity.

Exercise 1.11 [Blanchard and Fisher (1989), Problem …, pp. 289-290]:

Consumers maximise expected utility

(1.42) $$\max E\left[\sum_{t=0}^{T-1}\frac{1}{\gamma}e^{-\gamma c}\right]$$

subject to

(1.43) $$A_{t+1} = A_t + y_t - c_t$$

and

(1.44) $$y_t = y_{t-1} + e_t \qquad e_t \sim N\left(0, \sigma^2\right).$$

Derive the Euler equation marginal utility in period t = expected marginal utility in period t+1 for consumption under rational expectations analytically and plot the relationship between consumption in period t and period t+1 for different risk levels.

Solution: We can just use an ordinary Lagrangian to solve the system:

(1.45)
$$L = E\sum_{i=0}^{T-1}\left(-\frac{1}{\alpha}\right)e^{-\alpha C_t} + E\sum_{i=0}^{T-1}\lambda_i\left[A_t - A_{t+1} + Y_t - C_t\right]$$
$$= E\sum_{i=0}^{T-1}\left[\left(-\frac{1}{\alpha}\right)e^{-\alpha C_t} + \lambda_i\left[A_t - A_{t+1} + Y_t - C_t\right]\right].$$

The first order condition for the consumer at periods *t* and after is,

(1.46) $e^{-\alpha C_0} = \lambda_0,\ Ee^{-\alpha C_1} = \lambda_1,\ Ee^{-\alpha C_2} = \lambda_2,\ \ldots,$

The multipliers should be the same since there is no interest/discount rate for *A*. Thus

(1.47) $Ee^{-\alpha C_1} = \ldots = Ee^{-\alpha C_{T-1}} = e^{-\alpha C_0}$.

Alternatively, one can transform $A_{t+1} = A_t + Y_t - C_t$ into the following equation:

(1.48) $A_T = A_0 + \sum_{t=0}^{T-1} Y_t - \sum_{t=0}^{T-1} C_t$.

Since the consumer only lives for *T* periods, we need the transversality condition that $A_T = 0$. Thus, the life-long budget constraint becomes

(1.49) $A_0 + \sum_{t=0}^{T-1} Y_t = \sum_{t=0}^{T-1} C_t$.

The Lagrangian now becomes

(1.50) $L = E\sum_{i=0}^{T-1}\left(-\frac{1}{\alpha}\right)e^{-\alpha C_t} + \lambda\left[A_0 + \sum_{t=0}^{T-1} Y_t - \sum_{t=0}^{T-1} C_t\right].$

The first order conditions are the same as those we already have:

(1.51) $Ee^{-\alpha C_0} = Ee^{-\alpha C_1} = \ldots = Ee^{-\alpha C_{T-1}} = \lambda$.

Equation (1.49) indicates that *C* and *Y* have the same error terms/random walk.

(1.52) $C_t = EC_t + \varepsilon_t$,

where $\varepsilon_t \sim N(0, \sigma^2)$ and $C_t \sim N[EC_t, \sigma^2]$. From the first order conditions, for periods 0 and 1, we have

(1.53) $Ee^{-\alpha C_1} = e^{-\alpha C_0}$

The lognormal distribution (see Exercise 1.10) implies

(1.54) $E(e^{bx}) = e^{b\mu + \frac{b^2\sigma^2}{2}}$, if $x \sim N[\mu, \sigma^2]$.

Therefore, if $C_t \sim N[EC_t, \sigma^2]$, then

(1.55) $E\left[e^{-\alpha C_1}\right]=e^{-\alpha EC_1+\alpha^2\sigma^2/2}=e^{-\alpha C_0}$.

Taking logs on both sides,

(1.56) $-EC_1+\alpha\sigma^2/2=-C_0$

and substituting (1.52) into (1.56) gives

(1.57) $C_1=C_0+\dfrac{\alpha\sigma^2}{2}+\varepsilon_1$.

It is then straightforward to obtain

(1.58) $C_{t+1}=C_t+\dfrac{\alpha\sigma^2}{2}+\varepsilon_{t+1}$.

Rewriting the budget constraint leads to

(1.59) $A_t+\sum_{\tau=t}^{T-1}Y_\tau=\sum_{\tau=t}^{T-1}C_\tau$

Taking expectations yields

(1.60) $A_t+\sum_{\tau=t}^{T-1}E_tY_\tau=\sum_{\tau=t}^{T-1}E_tC_\tau$.

We know that

(1.61) $E_tY_{t+1}=...=E_tY_{T-1}=Y_t$.

And from (1.58) we have

(1.62) $E_tC_{t+s}=C_t+\dfrac{s\alpha\sigma^2}{2}$.

Substituting (1.61) and (1.62) into (1.60) yields

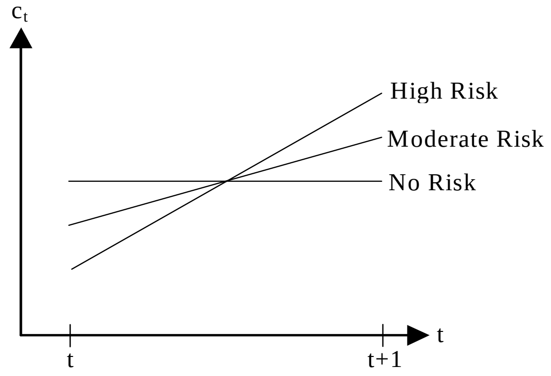
(1.63) $A_t+(T-t)Y_t=(T-t)C_t+\dfrac{\alpha\sigma^2}{2}\dfrac{(T-t)(T-t-1)}{2}$.

Rearranging finally gives

(1.64) $C_t=\left(\dfrac{1}{T-t}\right)A_t+Y_t-\dfrac{\alpha(T-t-1)\sigma^2}{4}$.

Equation (1.64) quantifies the sensitivity of consumption to the volatility level. The crux of the answer is that the relationship has a rather simple form – the slope is proportional to the variance of e_t. The proportionality factor is positive if utility is convex.

Figure 1.3: Consumption Profiles and Risk



With convex marginal utility, increasing uncertainty about consumption in period t+1 increases expected marginal utility. To restore the Euler equation, marginal utility today must go up and/or marginal utility tomorrow must fall. This is achieved by reducing consumption today, i.e. increasing precautionary saving. Put differently, the consumption path is steeper when agents are exposed to higher risk. Without risk (σ^2 = 0), the consumer wants to smooth consumption perfectly over time. Thus, the slope of the c_t path is zero. The higher the risk is, however, the stronger is the precautionary saving motive, so the slope becomes steeper as σ^2 rises.

<ins>Additional Reference:</ins>

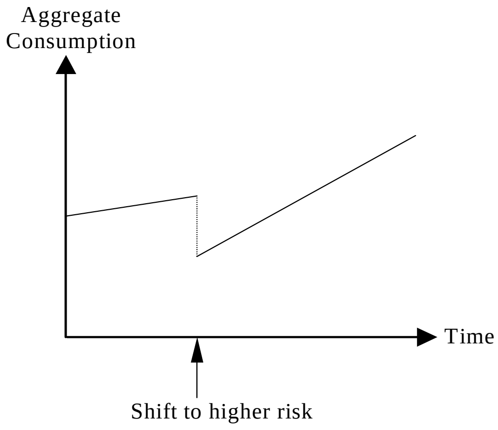
Blanchard, O.J. and S. Fisher (1989) *Lectures on Macroeconomics*, Cambridge (MIT Press).

Exercise 1.12:

Consider aggregate consumption over time. How does the consumption profile change when the (idiosyncratic) risk levels shifts to a higher level? Sketch the contour of consumption in a graph.

<ins>Solution:</ins> In general one should expect that prudent consumers save more when the risk increases. Higher savings imply higher future wealth and therefore increased risk is expected to cause the growth rate of consumption to rise. Furthermore, when risk is idiosyncratic, then an increase in risk increases the actual, not only the expected, consumption profile.

Figure 1.4: Aggregate Consumption and Increasing Risk

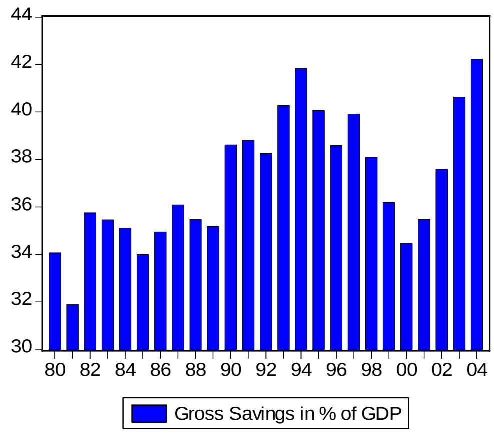


Exercise 1.13:

During the far-reaching economic transition process in Mainland China both the government and SOEs have, since the 1980s, reduced the social welfare net they provide. This expenditure has now become an individual responsibility: (a) in urban areas only about half the population is covered by basic health insurance, while in rural areas only about 20 percent of the population is covered by cooperative health insurance programmes; (b) the share of China's workforce in 2004 covered by unemployment insurance is only about 15 percent; (c) the basic government pension scheme is extremely limited and provides a pension equal to only 20 percent of average local wages; and finally (d) families are also responsible for a significant share of education expenditure, since government expenditure on education amounts to only 2 percent of GDP. What type of savings behaviour would you expect in light of the precautionary savings motive during the reform period? How has the Chinese savings rate developed over time?

Solution: Given the high level of uncertainty [high σ in equation (1.64)], the savings rate can be expected to be high in order to cope with life's uncertainties such as unemployment or illness. Furthermore families save because school fees are a large burden.

Figure 1.5: Gross savings in Percent of GDP in Mainland China, 1980 - 2004



Since the beginning of the 1980s, the Chinese gross saving rate has oscillated around very high levels by any international standard. By contrast, for example, in the U.S. consumers spent more than their disposible income in 2005, i.e the saving rate was slightly negative.
The difference between China´s consumption-to-GDP ratio and that in other countries is even higher when government consumption expenditures for education and healthcare are taken into account. While

some advanced countries, including Australia and Canada, have quite modest personal consumption shares, they often reflect that households receive substantial publicly provided goods like education and healthcare that are not included in measures of personal consumption in national accounts.

Table 1.1: Consumption Shares Across Countries (in percent of GDP, 2004)

	Personal Consumption	Government Consumption on Education and Healthcare	Σ
US	70	10	80
UK	65	12	77
Australia	60	11	71
Canada	56	7	63
Korea	51	5	56
France	56	6	62
Germany	57	6	63
Italy	60	12	72
Japan	57	5	62
India	67	4	70
China	41	3	44

Source: Aziz, J. (2006) "Rebalancing China´s Economy: What Does Growth Theory Tell US?", *IMF Working Paper WP/06/291*, Washington.

Exercise 1.14 (p. 35):

In the textbook the *subtractive* form of a habit consumption function is used, in which outer utility is derived from the *difference* between current consumption and the habit stock, $u(c,\chi)=(c-\chi)^{1-\gamma}/(1-\gamma)$ where χ is the level of habit. Alternatively, assume now that the habit consumption function is given by the *multiplicative* form

$$(1.65)\quad u(c,\chi)=\frac{\left(\frac{c}{\chi^{\phi}}\right)^{1-\gamma}}{1-\gamma}$$

(a) Interpret the parameter ϕ and (b) Demonstrate that the relative degree of risk aversion is still given by γ.

Solution: (a) If ϕ = 0, the model collapses to the standard CRRA modelling approach and habits are irrelevant. For ϕ = 1, consumers only care about how consumption compares to their habits, i.e. the level of consumption is irrelevant. (b) The relative degree of risk aversion is

$$(1.66)\quad \frac{-\left(-\frac{\gamma\chi^{-\phi}\left(c\chi^{-\phi}\right)^{-\gamma}}{c}\right)c}{\chi^{-\phi}\left(c\chi^{-\phi}\right)^{-\gamma}}=\gamma \ .$$

Exercise 1.15 (Blanchard and Fisher (1989), Problem 5, pp. 309-310):

Consider a consumer facing a stochastic taste shock. The consumer maximises the intertemporal linear-quadratic utility function

(1.67) $$E\left[\sum_{t=0}^{\infty}\frac{(a+e_t)C_t-bC_t^2}{(1+\rho)^t}\right]$$

subject to

(1.68) $$A_{t+1}=(1+r)(A_t+Y_t-C_t),$$

where $E[\cdot]$ is the expectations operator, C is consumption, Y is labour income, A is private wealth, r is the interest rate, and ρ is the discount rate. The disturbance term e_t captures stochastic taste shocks, shocks to marginal utility. For simplicity assume $\rho = r$.

(a) Derive the first order conditions.
(b) Assume that e_t follows the first-order autoregressive process $e_t=\alpha\, e_{t-1}+\varepsilon_t$ with $\varepsilon_t \sim N(0,\sigma^2)$ and $0 < \alpha < 1$. Characterise the resulting optimal consumption path.
(c) Alternatively assume that e_t follows the random walk $e_t=e_{t-1}+\varepsilon_t$ with $\varepsilon_t \sim N(0,\sigma^2)$. Again characterise the resulting optimal consumption path.

Solution:

Subtask (a): The Euler equation requires

(1.69) $$U'(C_t)=E[U'(C_{t+1})\,|\,t]$$

Since (1.69) is valid for all t, we have

(1.70) $$U'(C_t)=\frac{\partial U(C_t)}{\partial C_t}=(a+e_t)-2bC_t$$

Rearranging equation (1.70) leads to

(1.71) $$(a+e_t)-2bC_t=E[(a+e_{t+1})-2bC_{t+1}\,|\,t]$$

(1.72) $$(a+e_t)-2bC_t=a+E[e_{t+1}\,|\,t]-2bE[C_{t+1}\,|\,t]$$

(1.73) $$e_t-2bC_t=E[e_{t+1}\,|\,t]-2bE[C_{t+1}\,|\,t]$$

(1.74) $$\{E[C_{t+1}-C_t\,|\,t]\}=\frac{1}{2b}E[e_{t+1}-e_t\,|\,t].$$

Using backward iteration we finally have

(1.75) $$E[C_{t+1}\,|\,t]=C_t+\frac{1}{2b}E[e_{t+1}-e_t\,|\,t]$$

(1.76) $$E[E[C_{t+1}\,|\,t]\,|\,t-1]=E[C_t\,|\,t-1]+\frac{1}{2b}E[E[e_{t+1}-e_t\,|\,t]\,|\,t-1]$$

(1.77) $E[C_{t+1} | t-1] = E[C_t | t-1] + \frac{1}{2b} E[e_{t+1} - e_t | t-1]$

(1.78) $E[C_{t+1} | t-1] = C_{t-1} + \frac{1}{2b} E[e_t - e_{t-1} | t-1] + \frac{1}{2b} E[e_{t+1} - e_t | t-1]$

(1.79)

$$E[E[C_{t+1} | t-1] | t-2] = E[C_{t-1} | t-2] + \frac{1}{2b} E[E[e_t - e_{t-1} | t-1] | t-2] + \frac{1}{2b} E[E[e_{t+1} - e_t | t-1] | t-2]$$

(1.80) $E[C_{t+1} | t-2] = C_{t-2} + \frac{1}{2b} E[e_{t-1} - e_{t-2} | t-2] + \frac{1}{2b} E[e_t - e_{t-1} | t-2] + \frac{1}{2b} E[e_{t+1} - e_t | t-2]$

$\vdots$

(1.81) $E[C_{t+1} | 0] = C_0 + \frac{1}{2b} E[e_{t+1} | 0]$

Since $E[e_{t+1} | t] = 0$, we obtain $E[C_{t+1} | t] = C_t$, i.e. expected consumption is constant over time. The budget constraint then implies that consumption at each point in time is equal to lifetime wealth divided by the length of life.

Subtask (b): The dynamics of consumption can be derived in a conceptually straightforward manner, although the algebra becomes rather lengthy and tedious. The law of iterated projection yields

(1.82) $A_{t+1} = (1+r)(A_t + Y_t - C_t)$

(1.83) $A_{t+1} = (1+r)((1+r)(A_{t-1} + Y_{t-1} - C_{t-1}) + Y_t - C_t)$

(1.84) $A_{t+1} = (1+r)((1+r)((1+r)(A_{t-2} + Y_{t-2} - C_{t-2}) + Y_{t-1} - C_{t-1}) + Y_t - C_t)$

$\vdots$

(1.85) $A_{t+1} = (1+r)^{t+1} A_0 + \sum_{i=0}^{t} (1+r)^{t+1-i} (Y_i - C_i)$

As $t \to \infty$ and assuming the transversality condition that $\lim_{t\to\infty} \frac{A_t}{(1+r)^t} = 0$ gives us

(1.86) $\frac{A_\infty}{(1+r)^\infty} = A_0 + \sum_{i=0}^{\infty} (1+r)^{-i} (Y_i - C_i)$

(1.87) $A_0 + \sum_{i=0}^{\infty} (1+r)^{-i} Y_i = \sum_{i=0}^{\infty} (1+r)^{-i} C_i$

(1.88) $\sum_{i=0}^{\infty} (1+r)^{-i} E[C_i | 0] = A_0 + \sum_{i=0}^{\infty} (1+r)^{-i} E[Y_i | 0]$

Substituting (1.81) into (1.88) gives

(1.89) $$\sum_{i=0}^{\infty}(1+r)^{-i}\left(C_0+\frac{1}{2b}E[e_i\,|\,0]\right)=A_0+\sum_{i=0}^{\infty}(1+r)^{-i}E[Y_i\,|\,0]$$

(1.90) $$\left(\frac{r}{1+r}\right)^{-1}C_0+\frac{1}{2b}\sum_{i=0}^{\infty}(1+r)^{-i}E[e_i\,|\,0]=A_0+\sum_{i=0}^{\infty}(1+r)^{-i}E[Y_i\,|\,0]$$

(1.91) $$C_0=\left(\frac{r}{1+r}\right)\left[A_0+\sum_{i=0}^{\infty}(1+r)^{-i}E[Y_i\,|\,0]-\frac{1}{2b}\sum_{i=0}^{\infty}(1+r)^{-i}E[e_i\,|\,0]\right]$$

Since (1.91) holds in all periods, we can finally write consumption in period t as

(1.92) $$C_t=\left(\frac{r}{1+r}\right)\left[A_t+\sum_{i=0}^{\infty}(1+r)^{-i}E[Y_{t+i}\,|\,t]-\frac{1}{2b}\sum_{i=0}^{\infty}(1+r)^{-i}E[e_{t+i}\,|\,t]\right].$$

Thus the change in consumption from *t-1* to *t* is

(1.93) $$C_t-(1+r)C_{t-1}=\left(\frac{r}{1+r}\right)\left[\begin{array}{l}A_t+\sum_{i=0}^{\infty}(1+r)^{-i}E[Y_{t+i}\,|\,t]-\frac{1}{2b}\sum_{i=0}^{\infty}(1+r)^{-i}E[e_{t+i}\,|\,t]-\\(1+r)\left[A_{t-1}+\sum_{i=0}^{\infty}(1+r)^{-i}E[Y_{t+i-1}\,|\,t-1]-\frac{1}{2b}\sum_{i=0}^{\infty}(1+r)^{-i}E[e_{t+i-1}\,|\,t-1]\right]\end{array}\right]$$

(1.94) $$C_t-(1+r)C_{t-1}=\left(\frac{r}{1+r}\right)\left[\begin{array}{l}A_t-(1+r)A_{t-1}-(1+r)Y_{t-1}+\\\sum_{i=0}^{\infty}(1+r)^{-i}E[Y_{t+i}\,|\,t]-\frac{1}{2b}\sum_{i=0}^{\infty}(1+r)^{-i}E[e_{t+i}\,|\,t]-\\\left[\sum_{i=0}^{\infty}(1+r)^{-i}E[Y_{t+i}\,|\,t-1]-\frac{1+r}{2b}\sum_{i=0}^{\infty}(1+r)^{-i}E[e_{t+i-1}\,|\,t-1]\right]\end{array}\right]$$

Using the fact that $A_t-(1+r)A_{t-1}-(1+r)Y_{t-1}=-(1+r)C_{t-1}$ finally leads to

(1.95) $$C_t-C_{t-1}=\left(\frac{r}{1+r}\right)\left[\begin{array}{l}\sum_{i=0}^{\infty}(1+r)^{-i}\left(E[Y_{t+i}\,|\,t]-E[Y_{t+i}\,|\,t-1]\right)-\\\frac{1}{2b}\sum_{i=0}^{\infty}(1+r)^{-i}\left(E[e_{t+i}\,|\,t]-E[e_{t+i}\,|\,t-1]\right)\end{array}\right]+\frac{1}{2b}re_{t-1}.$$

The innovation e_t is assumed to follow the AR(1) process $e_t=\alpha e_{t-1}+\varepsilon_t$. Through similar manipulations as in Blanchard and Fisher (1989), p. 287 we have

(1.96)

$$\sum_{i=0}^{\infty}(1+r)^{-i}\left(E\left[\sum_{j=1}^{\infty}\alpha^j\,\varepsilon_{t+i-j}\,|\,t\right]-E\left[\sum_{j=1}^{\infty}\alpha^j\,\varepsilon_{t+i-j}\,|\,t-1\right]\right)=\sum_{i=0}^{\infty}(1+r)^{-i}\sum_{j=1}^{\infty}\alpha^j\left(E[\varepsilon_{t+i-j}\,|\,t]-E[\varepsilon_{t+i-j}\,|\,t-1]\right)$$

The expression in the last parenthesis is zero for $j > i$ because the shocks for innovations prior to *t-1* are identical (the additional information in t is irrelevant because the shocks are already known in *t-1*). Collecting the remaining terms gives us

$$(1.97)\quad \sum_{t=0}^{\infty}(1+r)^{-i}\alpha^{i}\varepsilon_t=\varepsilon_t\sum_{t=0}^{\infty}(1+r)^{-i}\alpha^{i}=\frac{\varepsilon_t}{\frac{\alpha}{(1+r)}}.$$

It is then straightforward to verify that the change in consumption in (1.95) simplifies to

$$(1.98)\quad C_t-C_{t-1}=\left(\frac{r}{1+r}\right)\left[\sum_{i=0}^{\infty}(1+r)^{-i}\left(E[Y_{t+i}\,|\,t]-E[Y_{t+i}\,|\,t-1]\right)\right]-\frac{1}{2b}\left(\left[\frac{\frac{r}{(1+r)}}{1-\frac{\alpha}{(1+r)}}\right]\varepsilon_t-re_{t-1}\right)$$

Equation (1.98) is intuitive. It implies that measured consumption is correlated, i.e. the change in consumption today provides us with some information as to what the change in consumption is likely to be tomorrow.

<u>Subtask (c):</u> When e_t is assumed to follow a random walk, then α = 1. Using (1.98), the following calculations then determine the dynamics of consumption:

$$(1.99)\quad C_t-C_{t-1}=\left(\frac{r}{1+r}\right)\left[\sum_{i=0}^{\infty}(1+r)^{-i}\left(E[Y_{t+i}\,|\,t]-E[Y_{t+i}\,|\,t-1]\right)\right]-\frac{1}{2b}\left(\left[\frac{\frac{r}{(1+r)}}{1-\frac{1}{(1+r)}}\right]\varepsilon_t-re_{t-1}\right)$$

$$(1.100)\quad C_t-C_{t-1}=\left(\frac{r}{1+r}\right)\left[\sum_{i=0}^{\infty}(1+r)^{-i}\left(E[Y_{t+i}\,|\,t]-E[Y_{t+i}\,|\,t-1]\right)\right]-\frac{1}{2b}\left(\varepsilon_t-re_{t-1}\right)$$

Using the fact that $e_t=e_{t-1}+\varepsilon_t=e_{t-2}+\varepsilon_{t-1}+\varepsilon_t=\cdots=\sum_{i=0}^{\infty}\varepsilon_{t-i}$ equation (1.100) can be rewritten as

$$(1.101)\quad C_t-C_{t-1}=\left(\frac{r}{1+r}\right)\left[\sum_{i=0}^{\infty}\frac{1}{(1+r)^{i}}\left(E[Y_{t+i}\,|\,t]-E[Y_{t+i}\,|\,t-1]\right)\right]-\frac{1}{2b}\left(\varepsilon_t-r\sum_{i=0}^{\infty}\varepsilon_{t-1-i}\right)$$

When the error term follows a random walk, successive changes in consumption are uncorrelated.

<u>Additional Reference:</u>

Blanchard, O.J. and S. Fisher (1989) *Lectures on Macroeconomics*, Cambridge (MIT Press).

Exercise 1.16 (p. 41):

Suppose consumers have the following utility function for durable goods (there are no non-durable goods like in Exercise 6 on p. 41 in the textbook):

(1.102) $$U_t = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^t u(S_t)$$

where

(1.103) $$S_t = (1-\delta) S_{t-1} + c_t \ ,$$

where S_t is the stock of durable goods, δ is the depreciation rate and c_t is the purchase of new durable goods. The accumulation constraint takes the form

(1.104) $$A_{t+1} = (1+r)(A_t + y_t - c_t) \ .$$

We assume that future wage incomes y_t are uncertain and consumer's have access to financial markets with no borrowing constraints.

(a) Set up the consumer's utility maximisation problem and obtain the Euler equation.

(b) Show that for a quadratic utility function and $\left(\frac{1+r}{1+\rho} \right) = 1$, the innovations in consumption follow a MA(1) process. Interpret the result.

<u>Solution:</u>

<u>Subtask (a):</u> The stock of durables in period *t* is

(1.105) $$S_t = \sum_{j=0}^{\infty} (1-\delta)^j c_{t-j} \ .$$

The consumer's problem can therefore be rewritten as

(1.106) $$\max_{c_t, A_{t+1}} E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^t u\left(\sum_{j=0}^{\infty} (1-\delta)^j c_{t-j} \right)$$

subject to the accumulation constraint (1.104). The first-order condition of the Lagrangian with respect to c_t gives

(1.107) $$\sum_{j=0}^{\infty} \left(\frac{1}{1+\rho} \right)^j (1-\delta)^j E_t u'(S_{t+j}) = \lambda_t \ ,$$

while the first-order condition for A_{t+1} is

(1.108) $$\lambda_t = (1+r) \lambda_{t+1}$$

where λ_t is the associated multiplier for the restriction. Thus we have

(1.109)

$$\sum_{j=0}^{\infty}\left(\frac{1}{1+\rho}\right)^{j}(1-\delta)^{j}E_t u'(S_{t+j})=(1+r)\lambda_{t+1}$$

$$\Leftrightarrow \sum_{j=0}^{\infty}\left(\frac{1}{1+\rho}\right)^{j}(1-\delta)^{j}E_t u'(S_{t+j})=\left(\frac{1+r}{1+\rho}\right)\sum_{j=0}^{\infty}\left(\frac{1}{1+\rho}\right)^{j}(1-\delta)^{j}E_t u'(S_{t+j+1})$$

$$\Leftrightarrow \left(\frac{1}{1+\rho}\right)(1-\delta)\sum_{j=0}^{\infty}\left(\frac{1}{1+\rho}\right)^{j}(1-\delta)^{j}E_t u'(S_{t+j+1})=\left(\frac{1}{1+\rho}\right)^{2}(1-\delta)(1+r)\sum_{j=0}^{\infty}\left(\frac{1}{1+\rho}\right)^{j}(1-\delta)^{j}E_t u'(S_{t+j+2})$$

$$\Leftrightarrow u'(S_t)=\left(\frac{1+r}{1+\rho}\right)E_t u'(S_{t+1}) \ .$$

Subtask (b): With a quadratic utility function and $r = \rho$, marginal utility is linear and therefore (1.109) becomes

(1.110) $$S_t = E_t S_{t+1} \Leftrightarrow S_{t+1} = S_t + \varepsilon_t \ ,$$

where $E_{t-1}\varepsilon_t = 0$. From (1.103) we have

(1.111) $$c_t = S_t - (1-\delta)S_{t-1} \Leftrightarrow c_{t-1} = S_{t-1} - (1-\delta)S_{t-2} \Leftrightarrow \Delta c_t = \Delta S_t - (1-\delta)\Delta S_{t-1},$$

where Δ is the first-difference operator. From (1.110) we know that $\Delta S_t = \varepsilon_t$ for all t and thus

(1.112) $$\Delta c_t = \varepsilon_t - (1-\delta)\varepsilon_{t-1} .$$

The intuition is clear. Let's assume that a positive shock $\varepsilon_t > 0$ occurs and therefore S_t is increasing. Then consumers don't have to buy as much durables in period t+1 as they still have the undepreciated part they have obtained in period t. Finally, in the special case of full depreciation within one period $(\delta = 1)$ we obtain the textbook result that consumption follows a random walk, i.e. $\Delta c_t = \varepsilon_t$.

II. Software Tools

Software Exercise 1.1 [Alessie and Lusardi (1997)]:

According to Alessie and Lusardi (1997), the consumer maximizes the utility function

(1.113) $$\max E_t \sum_{\tau=t}^{\infty} -\frac{1}{\theta}\frac{e^{-\theta(c_\tau - \gamma c_{\tau-1})}}{(1+\rho)^{t-\tau}} ,$$

subject to the following intertemporal budget constraint

(1.114) $$\sum_{\tau=t}^{\infty}\frac{c_\tau}{(1+r)^{\tau-t}} = (1+r)A_{t-1} + \sum_{\tau=t}^{\infty}\frac{y_\tau}{(1+r)^{\tau-t}} ,$$

where c is the consumption, y is the non-capital income, A_{t-1} is the non-human wealth at the end of period t-1, θ is a positive parameter, ρ is the time preference rate, r is real interest rate, γ is a positive parameter related to consumption at the previous period and $0 < \gamma < 1$. Habitual behaviour is usually slow to change and therefore past behaviour exercises influences over current behaviour that is "habitual". The solution of the consumption process is determined by

$$c_t = \frac{\gamma}{(1+r)} c_{t-1} + \left(1 - \frac{\gamma}{(1+r)}\right) Y_{pt} - \frac{r}{(1+r)} \sum_{\tau=t+1}^{\infty} (1+r)^{t-\tau} \sum_{j=t+1}^{\tau} \Gamma_{j-1} , \tag{1.115}$$

where

$$\Gamma_{j-1} = \frac{1}{\theta} \ln E_{j-1} \exp\left(-\theta \psi * w_j\right), \tag{1.116}$$

$$\psi^* = \left(1 - \frac{\gamma}{(1+r)}\right) \frac{r}{(1+r)} \sum_{t=0}^{\infty} \psi_t (1+r)^{-t} , \tag{1.117}$$

$$Y_{pt} = \frac{r}{(1+r)} \left((1+r) A_{t-1} + \sum_{\tau=1}^{\infty} (1+r)^{t-\tau} E_t y_\tau \right) \tag{1.118}$$

and

$$E_t y_\tau - E_{t-1} y_\tau = \psi_{\tau-t} w_t , \quad \psi_0 = 1 \tag{1.119}$$

where w_t is an innovation disturbance. Non-capital income is assumed to follow a stochastic process with MA(1) shocks,

$$y_t = y_{t-1} + w_t , \tag{1.120}$$

$$w_t = e_t + \alpha e_{t-1} . \tag{1.121}$$

Thus,

$$y_{t+1} = y_t + e_{t+1} + \alpha e_t = y_{t-1} + e_t + e_{t+1} + \alpha(e_{t-1} + e_t), \tag{1.122}$$

$$y_{t+2} = y_{t+1} + e_{t+2} + \alpha e_{t+1} = y_{t-1} + e_t + e_{t+1} + e_{t+2} + \alpha(e_{t-1} + e_t + \alpha e_{t+1}). \tag{1.123}$$

From (1.119), we have

$$\psi_0 = 1 = \frac{E_t y_t - E_{t-1} y_t}{w_t} . \tag{1.124}$$

Substituting (1.122) into (1.119) yields

$$\psi_1 = \frac{E_t y_{t+1} - E_{t-1} y_{t+1}}{w_t}$$

(1.125) $$= \frac{E_t\left[y_{t-1} + e_t + e_{t+1} + \alpha\left(e_{t-1} + e_t\right)\right] - E_{t-1}\left[y_{t-1} + e_t + e_{t+1} + \alpha\left(e_{t-1} + e_t\right)\right]}{w_t}$$

$$= \frac{E_t\left[y_{t-1}\right] - E_{t-1}\left[y_{t-1}\right]}{w_t} = 1.$$

Thus,

(1.126) $$\psi_j = 1, \quad j = 0, \ldots, \infty .$$

Substituting (1.126) into (1.115) yields

(1.127) $$\psi^* = \left(1 - \frac{\gamma}{(1+r)}\right)\frac{r}{(1+r)}\sum_{t=0}^{\infty}\psi_t (1+r)^{-t} = \left(1 - \frac{\gamma}{(1+r)}\right)\frac{r}{(1+r)}\sum_{t=0}^{\infty}(1+r)^{-t} ,$$

(1.128) $$\psi^* = \left(1 - \frac{\gamma}{(1+r)}\right)\frac{r}{(1+r)}\left(\frac{1+r}{r}\right) = \left(1 - \frac{\gamma}{(1+r)}\right),$$

since $\sum_{t=0}^{\infty}(1+r)^{-t} = \left(\frac{1+r}{r}\right)$.

Substituting (1.128) into (1.116) yields

(1.129) $$\Gamma_{j-1} = \frac{1}{\theta}\ln E_{j-1}\exp\left(-\theta\left(1 - \frac{\gamma}{(1+r)}\right)\left(e_t + \alpha e_{t-1}\right)\right).$$

The lognormal distribution (see Exercise 1.10) implies $E\left(e^{bx}\right) = e^{b\,\mu + \frac{b^2\sigma^2}{2}}$, if $x \sim N\left[\mu, \sigma^2\right]$. Therefore, if we assume that $e_t \sim N\left[0, \sigma_e^2\right]$ and i.i.d., then we obtain

(1.130) $$\Gamma_{j-1} = \frac{1}{\theta}\ln\exp\left(\theta^2\left(1 - \frac{\gamma}{(1+r)}\right)^2\left(\frac{\sigma_e^2}{2} + \frac{\alpha^2\sigma_e^2}{2}\right)\right) = \theta\left(1 - \frac{\gamma}{(1+r)}\right)^2\left(1 + \alpha^2\right)\frac{\sigma_e^2}{2} .$$

(1.122) and (1.123) imply that $E_t y_\tau = y_t$. Thus equation (1.118) becomes

(1.131) $$Y_{pt} = \frac{r}{(1+r)}\left((1+r)A_{t-1} + y_t\sum_{\tau=t}^{\infty}(1+r)^{t-\tau}\right),$$

(1.132) $$Y_{pt} = \frac{r}{(1+r)}\left((1+r)A_{t-1} + y_t\left(\frac{1+r}{r}\right)\right) = rA_{t-1} + y_t .$$

Substituting (1.130) and (1.132) back into (1.115) gives

(1.133) $$c_t = \frac{\gamma}{(1+r)} c_{t-1} + \left(1 - \frac{\gamma}{(1+r)}\right)(rA_{t-1} + y_t) - \frac{r}{(1+r)}\theta\left(1 - \frac{\gamma}{(1+r)}\right)^2 (1+\alpha^2)\frac{\sigma_e^2}{2} \sum_{\tau=t+1}^{\infty} \frac{(\tau - t)}{(1+r)^{\tau-t}}.$$

The term of $P = \sum_{\tau=t+1}^{\infty} \frac{(\tau - t)}{(1+r)^{\tau-t}}$ can be solved by the following procedure:

(1.134) $$P = \frac{1}{1+r} + \frac{2}{(1+r)^2} + \frac{3}{(1+r)^3} + \ldots,$$

(1.135) $$(1+r)P = 1 + \frac{2}{(1+r)} + \frac{3}{(1+r)^2} + \ldots.$$

(1.134) – (1.135) yields

(1.136) $$rP = 1 + \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \ldots. = \frac{1+r}{r}.$$

Thus,

(1.137) $$P = \sum_{\tau=t+1}^{\infty} \frac{(\tau - t)}{(1+r)^{\tau-t}} = \frac{1+r}{r^2}.$$

Substituting (1.137) into (1.133) finally gives

(1.138) $$c_t = \frac{\gamma}{(1+r)} c_{t-1} + \left(1 - \frac{\gamma}{(1+r)}\right)(rA_{t-1} + y_t) - \frac{1}{r}\theta\left(1 - \frac{\gamma}{(1+r)}\right)^2 (1+\alpha^2)\frac{\sigma_e^2}{2}.$$

Matlab simulations:

There are two Matlab files for equation (1.138):

- `HabitConsumption_a.m`: Only one shock at period t for e_t;
- `HabitConsumption_b.m`: Continuous shocks for e_t;

The results of `HabitConsumption_a.m`:

Students need to change/modify the following input data

```
% beginning of input data
Ctm1 = 1.0;  % C(t-1)
Ytm1 = 1.05;  % Y(t-1)
Atm1 = 0.6;  % A(t-1)
r    = 0.1; % real interest rate, annual rate
gamma = 0.6;
theta = 0.5;
sigma = 0.08; % annual % s.d. of Y(t)
alpha = 0.4;
period = 50; % The periods of C(t) to be computed
shock = -2.0; % The ONLY shock in terms of Normal distribution: ~N(0,1)
% end of input data
```

For period *t*, the Matlab file computes the consumption at *t* via (1.138), based on the input data. Equation (1.114) implies that

$$(1.139)\quad A_t = (1+r)A_{t-1} + y_t - c_t,$$

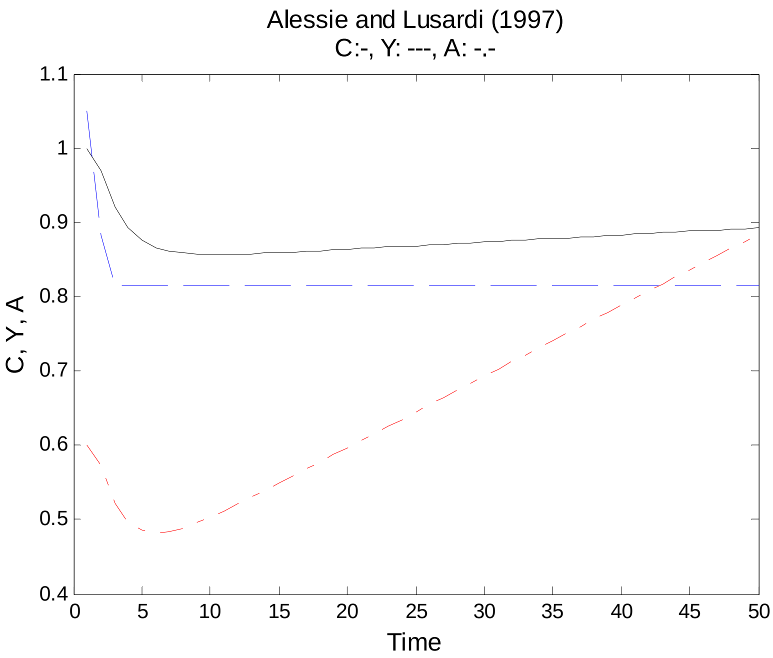
At the end of period *t*, the new wealth is computed based on equation (1.139). For period *t*+1, and periods after, $E[y_{t+1}] = y_t$ due to the expected values of y process.

```
Input data:
      r = 0.1000,  sigma =  0.080, gamma =  0.600
  theta =  0.500,      T = 50.000  e(t) = -0.168
 C(t-1) =  1.000, A(t-1) =  0.600, Y(t) = 1.0500

Results:
     t      C(t)     A(t)     Y(t)
     t- 1   1.00000  0.60000  1.05000
     t+ 0   0.97003  0.57197  0.88200
     t+ 1   0.92950  0.53126  0.83160
     t+ 2   0.90555  0.51044  0.83160
     t+ 3   0.89153  0.50155  0.83160
     t+ 4   0.88349  0.49982  0.83160
     t+ 5   0.87902  0.50238  0.83160
     t+ 6   0.87670  0.50753  0.83160
     t+ 7   0.87566  0.51421  0.83160
     t+ 8   0.87540  0.52183  0.83160
     t+ 9   0.87561  0.53000  0.83160
     t+10   0.87609  0.53851  0.83160
     t+11   0.87674  0.54722  0.83160
     t+12   0.87749  0.55605  0.83160
     t+13   0.87830  0.56495  0.83160
     t+14   0.87915  0.57389  0.83160
     t+15   0.88002  0.58286  0.83160
     t+16   0.88090  0.59184  0.83160
     t+17   0.88179  0.60084  0.83160
     t+18   0.88268  0.60984  0.83160
     t+19   0.88358  0.61884  0.83160
     t+20   0.88448  0.62785  0.83160
     t+21   0.88538  0.63685  0.83160
     t+22   0.88628  0.64586  0.83160
     t+23   0.88718  0.65487  0.83160
     t+24   0.88808  0.66388  0.83160
     t+25   0.88898  0.67288  0.83160
     t+26   0.88988  0.68189  0.83160
     t+27   0.89078  0.69090  0.83160
     t+28   0.89168  0.69991  0.83160
     t+29   0.89258  0.70892  0.83160
     t+30   0.89348  0.71793  0.83160
     t+31   0.89438  0.72693  0.83160
     t+32   0.89529  0.73594  0.83160
     t+33   0.89619  0.74495  0.83160
     t+34   0.89709  0.75396  0.83160
     t+35   0.89799  0.76297  0.83160
     t+36   0.89889  0.77198  0.83160
     t+37   0.89979  0.78098  0.83160
     t+38   0.90069  0.78999  0.83160
     t+39   0.90159  0.79900  0.83160
     t+40   0.90249  0.80801  0.83160
     t+41   0.90339  0.81702  0.83160
     t+42   0.90429  0.82602  0.83160
     t+43   0.90519  0.83503  0.83160
     t+44   0.90610  0.84404  0.83160
     t+45   0.90700  0.85305  0.83160
     t+46   0.90790  0.86206  0.83160
     t+47   0.90880  0.87107  0.83160
```

```
t+48  0.90970  0.88007  0.83160
t+49  0.91060  0.88908  0.83160
t+50  0.91150  0.89809  0.83160
```

The Matlab program also automatically provides us with a time-series figure for c(t), y(t), A(t):



The results of `HabitConsumption_b.m`:

The input data needed for `HabitConsumption_b.m` is almost completely the same as `HabitConsumption_a.m`, except that the Matlab file simulates the shocks of $e(t)$ by the standard normal distribution generator `randn(1)`, which is based on $N \sim (0,1)$. After inserting the input data

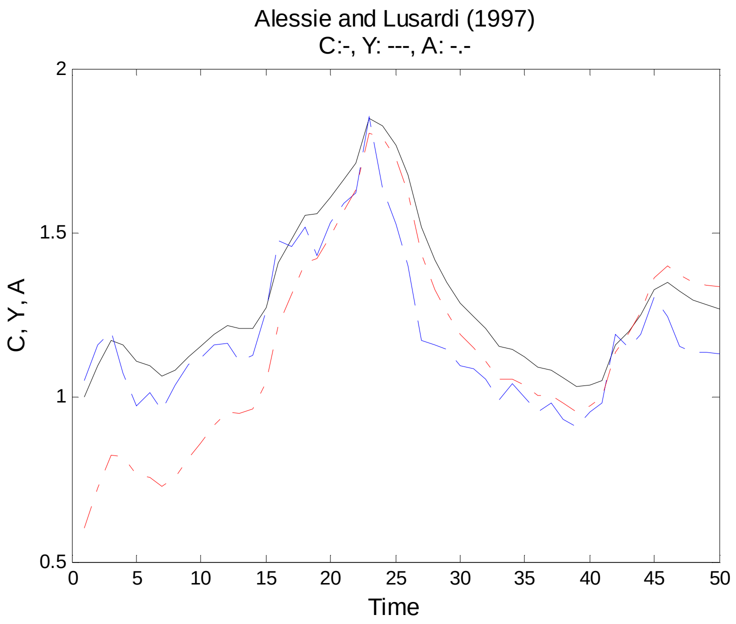
```
% beginning of input data
Ctm1 = 1.0;  % C(t-1)
Ytm1 = 1.1;  % Y(t-1)
Atm1 = 0.6;    % A(t-1)
r    = 0.1; % real interest rate, annual rate
gamma = 0.6;
theta = 0.5;
sigma = 0.1; % annual % s.d. of Y(t)
alpha = 0.7;
period = 50; % The periods of C(t) to be computed
% end of input data
```

the Matlab file gives the following results and figure:

```
Input data:
      r = 0.1000,   sigma =  0.080, gamma =  0.600
  theta =  0.500,       T = 50.000
 C(t-1) =  1.000, A(t-1) =  0.600, Y(t) = 1.0500

Results:
     t     C(t)     A(t)     Y(t)
     t- 1  1.00000  0.60000  1.05000
     t+ 0  1.09534  0.72285  1.15819
     t+ 1  1.17203  0.82331  1.20022
     t+ 2  1.16001  0.81733  1.07169
     t+ 3  1.10823  0.76362  0.97278
```

t+ 4	1.09559	0.75689	1.01251
t+ 5	1.06551	0.72924	0.96217
t+ 6	1.08241	0.75795	1.03821
t+ 7	1.12141	0.81318	1.10084
t+ 8	1.15365	0.86031	1.11946
t+ 9	1.19094	0.91350	1.15810
t+10	1.21701	0.95322	1.16538
t+11	1.20735	0.95005	1.10887
t+12	1.21105	0.96294	1.12893
t+13	1.27459	1.04761	1.26297
t+14	1.40951	1.21796	1.47509
t+15	1.48318	1.31480	1.45823
t+16	1.55549	1.41001	1.51922
t+17	1.55962	1.42340	1.43202
t+18	1.60719	1.48892	1.53037
t+19	1.66391	1.56542	1.59152
t+20	1.71131	1.63074	1.62009
t+21	1.84613	1.80096	1.85327
t+22	1.82768	1.78725	1.63388
t+23	1.76767	1.72367	1.52536
t+24	1.67446	1.62026	1.39868
t+25	1.51556	1.43801	1.17128
t+26	1.41604	1.32703	1.16126
t+27	1.35049	1.25681	1.14757
t+28	1.28788	1.19011	1.09550
t+29	1.24732	1.14987	1.08806
t+30	1.20747	1.11050	1.05311
t+31	1.15552	1.05658	0.99056
t+32	1.14774	1.05568	1.04118
t+33	1.12526	1.03715	1.00115
t+34	1.09041	1.00377	0.95332
t+35	1.08323	1.00358	0.98266
t+36	1.05746	0.98109	0.93461
t+37	1.03062	0.95733	0.90875
t+38	1.03553	0.97165	0.95412
t+39	1.05251	1.00046	0.98415
t+40	1.15758	1.13499	1.19206
t+41	1.20133	1.19592	1.14876
t+42	1.24807	1.26044	1.19300
t+43	1.32734	1.36400	1.30485
t+44	1.34921	1.39869	1.24750
t+45	1.32028	1.37241	1.15413
t+46	1.29542	1.35101	1.13677
t+47	1.28147	1.34271	1.13808
t+48	1.27010	1.33750	1.13062
t+49	1.19831	1.25979	0.98684
t+50	1.19108	1.25955	1.06487



Please note that the results and figure should be different every time the Matlab file is run due to the randomness of the shocks.

Further Software Exercises:

Show the impact of changes in the following parameters on the time-series *C(t)* and *A(t)* by running the Matlab file `HabitConsumption_a.m`, and provide an economic interpretation of the results:

1. The habit persistence parameter γ ,
2. the real interest rate *r*,
3. the standard deviation σ ,
4. a random walk vs MA(1), and
5. the θ parameter.

Additional Reference:

Alessie, R. and A. Lusardi (1997) "Consumption, Saving and Habit Formation, *Economics Letters* 55, 103-108.